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# Iteration-free *PDL* with storing, recovering and parallel composition: a complete axiomatization

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## Abstract

We devote this article to the axiomatization/completeness of  $PRSPDL_0$ —a variant of iteration-free *PDL* with parallel composition. Our results are based on the following: although the program operation of parallel composition is not modally definable in the ordinary language of *PDL*, it becomes definable in a modal language strengthened by the introduction of propositional quantifiers. Instead of using axioms to define the program operation of parallel composition in the language of *PDL* enlarged with propositional quantifiers, we add an unorthodox rule of proof that makes the canonical model standard for the program operation of parallel composition and we use large programs for the proof of the Truth Lemma.

*Keywords:* Iteration-free *PDL*, parallel composition, axiomatisation, completeness, expressivity.

## 1 Introduction

Propositional dynamic logic (*PDL*) is an applied logic *par excellence*. Designed for reasoning about the behaviour of programs [12, 15, 17], its syntax is based on the idea of associating with each program  $\alpha$  of some programming language the modal operator  $[\alpha]$ , formulas  $[\alpha]\phi$  being read ‘all executions of  $\alpha$  from the current state lead to a state where  $\phi$  holds’. Syntactically, *PDL* is a modal logic with a structure in the set of modal operators: sequential composition  $(\alpha; \beta)$  of programs  $\alpha$  and  $\beta$  corresponds to the composition of the accessibility relations  $R(\alpha)$  and  $R(\beta)$ ; test  $\phi?$  on formula  $\phi$  corresponds to the partial identity relation in the subsets of the Kripke models in which the formula  $\phi$  is true; iteration  $\alpha^*$  corresponds to the reflexive and transitive closure of  $R(\alpha)$ . The problem with *PDL* is that the states of the Kripke models in which formulas are evaluated have no internal structure. However, in the field of applied logics, formalisms with which one can cope with structured data are needed.

In separation logics, the formula construct  $(\cdot \circ \cdot)$  of separating conjunction, formulas  $(\phi \circ \psi)$  being read ‘the memory model can be split into 2 disjoint models respectively satisfying  $\phi$  and  $\psi$ ’, and the formula construct  $(\cdot -\circ \cdot)$  of adjoint implication, formulas  $(\phi -\circ \psi)$  being read ‘if the memory model is extended with a model satisfying  $\phi$ , the resulting model satisfies  $\psi$ ’, are added to the standard Boolean constructs [8, 10, 18, 19]. The propositional dynamic logic with storing, recovering and parallel composition (*PRSPDL*) introduced by Benevides *et al.* [5], is a separation-based logic too. Its Kripke models are structured by means of a function  $\star$ : the state  $x$  belongs to the result of applying the function  $\star$  to the states  $y, z$  iff  $x$  can be separated in a first part  $y$  and a second part  $z$ . Its syntax is obtained from the *PDL*-syntax by adding the program construct  $(\cdot \parallel \cdot)$  of parallel composition, the storing programs  $s_1$  and  $s_2$  and the recovering programs  $r_1$  and  $r_2$ . Among the separation logics considered in [8, 10, 18, 19], the one developed by Collinson and Pym [10] is the sole separation logic

to include modal operators. As a result, it seems to be the one that is the more similar to *PRSPDL*. Nevertheless, Collinson and Pym consider neither the construct of sequential composition nor the construct of test on formula. Moreover, their construct of parallel composition is associative and commutative whereas *PRSPDL*'s one is not.

In this variant of *PDL*, parallel composition  $(\alpha \parallel \beta)$  of programs  $\alpha$  and  $\beta$  corresponds to the fork  $R(\alpha) \nabla R(\beta)$  of the accessibility relations  $R(\alpha)$  and  $R(\beta)$ . More precisely, whenever  $x$  and  $y$  are related via  $R(\alpha)$  and  $z$  and  $t$  are related via  $R(\beta)$ , states in  $x * z$  and states in  $y * t$  are related via  $R(\alpha) \nabla R(\beta)$ . See also [13, Chapter 1] for another interpretation of the fork of two accessibility relations. About  $s_1$  and  $s_2$ ,  $x$  is related, by  $R(s_1)$ , to the states in  $x * z$  and, by  $R(s_2)$ , to the states in  $z * x$ : to execute  $s_1$  and  $s_2$  beginning from the state  $x$  is to store  $x$  as the first part of a larger state and as the second part of a larger state. As for  $r_1$  and  $r_2$ , the states in  $x * z$ , by  $R(r_1)$ , and the states in  $z * x$ , by  $R(r_2)$ , are related to  $x$ : to execute  $r_1$  and  $r_2$  ending in the state  $x$  is to recover  $x$  as the first part of the current state and as the second part of the current state. Hence,  $s_1$ ,  $s_2$ ,  $r_1$  and  $r_2$  enable us to view states as pairs of states. In the frames considered by Benevides *et al.* [5], if  $x$  belongs both to the result of applying  $\star$  to  $y, z$  and to the result of applying  $\star$  to  $t, u$ ,  $y = t$  and  $z = u$ . After their paper, seeing that the modal operator  $[\alpha \parallel \beta]$  cannot be defined by means of the modal operators  $[\alpha]$  and  $[\beta]$ , the problem of finding a complete axiomatization of *PRSPDL* remained open. The difficulty of this problem lies in the fact that the modal operator  $[\alpha \parallel \beta]$  cannot be defined by means of the modal operators  $[\alpha]$  and  $[\beta]$ .

The purpose of this article is the axiomatization/completeness of the set of all iteration-free formulas determined by the class of all separated frames. We attack the problem by a method based on an unorthodox rule of proof that makes the canonical model standard for the program operation of parallel composition. Our method follows the line of reasoning developed for *PDL* with intersection of programs [3, 4]. Nevertheless, this line of reasoning could not be used as it was, seeing that parallel composition of programs and intersection of programs are not interdefinable in the ordinary language of *PDL*. As a result, we had to redefine the fundamental notion of admissible forms used in [3, 4] to prepare the ground for Lemma 11, our main result. Sections 2 and 3 present the syntax and the semantics of *PRSPDL*<sub>0</sub>—a variant of iteration-free *PDL* with parallel composition. Expressivity results and definability results are given in Sections 4 and 5. Sections 6 and 7 contain an axiomatization of *PRSPDL*<sub>0</sub> and a proof of its completeness.

## 2 Syntax

This section presents the syntax of *PRSPDL*<sub>0</sub>. As usual, we will follow the standard rules for omission of the parentheses. The set *PR* of all programs and the set *FO* of all formulas are inductively defined as follows:

- $\alpha, \beta \longrightarrow a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \parallel \beta)$ ;
- $\phi, \psi \longrightarrow p \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$ ;

where  $a$  ranges over a countably infinite set *AP* of atomic programs and  $p$  ranges over a countably infinite set *AF* of atomic formulas. We will use  $\alpha, \beta, \dots$  for programs and  $\phi, \psi, \dots$  for formulas. Programs of the form  $\phi?$  will be called ‘tests’, programs  $s_1$  and  $s_2$  will be called ‘storing constructs’ and programs  $r_1$  and  $r_2$  will be called ‘recovering constructs’. The other Boolean constructs for formulas are defined as usual. Let  $\langle \alpha \rangle \phi ::= \neg[\alpha]\neg\phi$ .

EXAMPLE: If  $\alpha, \beta$  are programs and  $\phi, \psi$  are formulas,  $\langle \alpha \parallel \beta \rangle \phi \rightarrow \langle r_1; \alpha; s_1 \rangle (\phi \wedge \psi) \vee \langle r_2; \beta; s_2 \rangle (\phi \wedge \neg\psi)$  is a formula.

It is well worth noting that programs and formulas are finite strings of symbols coming from a countable alphabet. It follows that there are countably many programs and countably many formulas. Obviously, programs are built up from atomic programs, tests, storing constructs and recovering constructs by means of the constructs  $(\cdot; \cdot)$  and  $(\cdot \parallel \cdot)$ . The construct  $(\cdot; \cdot)$  comes from the class of algebras of binary relations [20]: the program  $\alpha; \beta$  first executes  $\alpha$  and secondly executes  $\beta$ . As for the construct  $(\cdot \parallel \cdot)$ , it comes from the class of proper fork algebras [13, Chapter 1]: the program  $\alpha \parallel \beta$  performs a kind of parallel execution of  $\alpha$  and  $\beta$ . The construct  $[\cdot]$  comes from the language of *PDL* [12, 17]: the formula  $[\alpha]\phi$  says that ‘every execution of  $\alpha$  from the present state leads to a state bearing the information  $\phi$ ’. Let  $\alpha(\phi_1?, \dots, \phi_n?)$  be a program with  $(\phi_1?, \dots, \phi_n?)$  a sequence of some of its tests. The result of the replacement of  $\phi_1?, \dots, \phi_n?$  in their places with  $\psi_1?, \dots, \psi_n?$  is another program which will be denoted  $\alpha(\psi_1?, \dots, \psi_n?)$ .

EXAMPLE: If  $\alpha, \beta, \gamma$  are programs and  $\phi, \psi$  are formulas, the result of the replacement of  $\phi?$  in its place in the program  $\alpha \parallel (\beta; \phi? \gamma)$  with  $\psi?$  is the program  $\alpha \parallel (\beta; \psi? \gamma)$ .

Let  $f$  be the function from the set of all programs into itself inductively defined as follows:

- $f(a) = a$ ;
- $f(\phi?) = \phi?$ ;
- $f(s_1) = s_1$ ;
- $f(s_2) = s_2$ ;
- $f(r_1) = r_1$ ;
- $f(r_2) = r_2$ ;
- $f(\alpha; \beta) = f(\alpha); \top?; f(\beta)$ ;
- $f(\alpha \parallel \beta) = (\top?; f(\alpha); \top?) \parallel (\top?; f(\beta); \top?)$ .

EXAMPLE: By definition,  $f(a \parallel (b; c)) = (\top?; a; \top?) \parallel (\top?; b; \top?; c; \top?)$ . The function  $f$  will be of use to us when we define the axiomatization of *PRSPDL*<sub>0</sub>, in particular the formula (A14). The set *PAR* of all parametrized programs and the set *ADM* of all admissible forms are inductively defined as follows:

- $\check{\alpha}, \check{\beta} \longrightarrow \neg\check{\phi}? | (\check{\alpha}; \check{\beta}) | (\check{\alpha} \parallel \check{\beta}) | (\check{\alpha} \parallel \check{\beta})$ ;
- $\check{\phi}, \check{\psi} \longrightarrow \sharp | [\check{\alpha}] \perp$ ;

where  $\sharp$  is a new atomic formula and  $\alpha, \beta$  range over *PR*. We will use  $\check{\alpha}, \check{\beta}, \dots$  for parametrized actions and  $\check{\phi}, \check{\psi}, \dots$  for admissible forms.

EXAMPLE: If  $\alpha, \beta, \gamma$  are programs,  $\alpha \parallel (\beta; \neg\sharp?; \gamma)$  is a parametrized program and  $[\alpha \parallel (\beta; \neg\sharp?; \gamma)] \perp$  is an admissible form.

Let  $\alpha(\phi?)$  be a program with  $\phi?$  some of its tests. The result of the replacement of  $\phi?$  in its place with a parametrized program  $\check{\beta}$  is a parametrized program which will be denoted  $\alpha(\check{\beta})$ .

EXAMPLE: If  $\alpha, \beta, \gamma$  are programs,  $\phi$  is a formula and  $\check{\delta}$  is a parametrized program, the result of the replacement of  $\phi?$  in its place in the program  $\alpha \parallel (\beta; \phi?; \gamma)$  with  $\check{\delta}$  is the parametrized program  $\alpha \parallel (\beta; \check{\delta}; \gamma)$ .

It is well worth noting that parametrized actions and admissible forms are finite strings of symbols coming from a countable alphabet. It follows that there are countably many parametrized actions and countably many admissible forms. Remark that in each expression *exp* (either a parametrized action, or an admissible form),  $\sharp$  has a unique occurrence. The result of the replacement of  $\sharp$  in its place in

$e\check{x}p$  with a formula  $\phi$  is an expression (either a program, or a formula) which will be denoted  $e\check{x}p(\phi)$ . It is inductively defined as follows:

- $\neg\check{\psi}(\phi) = \neg\check{\psi}(\phi)?$ ,
- $(\check{\alpha}; \check{\beta})(\phi) = \check{\alpha}(\phi); \check{\beta}$ ,
- $(\check{\alpha}; \check{\beta})(\phi) = \check{\alpha}; \check{\beta}(\phi)$ ,
- $(\check{\alpha} \parallel \check{\beta})(\phi) = \check{\alpha}(\phi) \parallel \check{\beta}$ ,
- $(\check{\alpha} \parallel \check{\beta})(\phi) = \check{\alpha} \parallel \check{\beta}(\phi)$ ,
- $\sharp(\phi) = \phi$ ,
- $[\check{\alpha}] \perp(\phi) = [\check{\alpha}(\phi)] \perp$ .

EXAMPLE: If  $e\check{x}p$  is the parametrized programs  $\alpha \parallel (\beta; \neg\sharp?; \gamma)$ ,  $e\check{x}p(\phi)$  is the program  $\alpha \parallel (\beta; \neg\phi?; \gamma)$  and if  $e\check{x}p$  is the admissible form  $[\alpha \parallel (\beta; \neg\sharp?; \gamma)] \perp$ ,  $e\check{x}p(\phi)$  is the formula  $[\alpha \parallel (\beta; \neg\phi?; \gamma)] \perp$ .

We will use parametrized actions and admissible forms when we define the axiomatization of  $PRSPDL_0$ , in particular the rule of proof (*FOR*).

### 3 Semantics

This section presents the semantics of  $PRSPDL_0$ . A frame is a 3-tuple  $\mathcal{F} = (W, R, \star)$  where  $W$  is a nonempty set of states,  $R$  is a function from the set of all atomic programs into the set of all binary relations between states and  $\star$  is a function from the set of all pairs of states into the set of all sets of states. We will use  $x, y, \dots$  for states. The set  $W$  of states in a frame  $\mathcal{F} = (W, R, \star)$  is to be regarded as the set of all possible states in a computation process, the function  $R$  from the set of all atomic programs into the set of all binary relations between states associates with each atomic program  $a$  the binary relation  $R(a)$  on  $W$  with  $xR(a)y$  meaning that ‘ $y$  can be reached from  $x$  by performing atomic program  $a$ ’ and the function  $\star$  from the set of all pairs of states into the set of all sets of states associates with each pair  $(x, y)$  of states the subset  $x\star y$  of  $W$  with  $z \in x\star y$  meaning that ‘ $z$  is a possible combination of  $x$  and  $y$ ’. We shall say that a frame  $\mathcal{F} = (W, R, \star)$  is functional iff for all  $x, y, z \in W$ , if  $xR(a)y$  and  $xR(a)z$ ,  $y = z$  for every program variable  $a$ . For all  $z \in W$ , let  $\star(z) = \{(x, y): z \in x\star y\}$ .

Now,  $card(\cdot)$  denoting the cardinality function, we consider the following classes of frames:

- separated frames, i.e. frames  $\mathcal{F} = (W, R, \star)$  such that for all  $x \in W$ ,  $card(\star(x)) \leq 1$ ;
- rich frames, i.e. frames  $\mathcal{F} = (W, R, \star)$  such that for all  $x \in W$ ,  $card(\star(x)) \geq 1$ ;
- deterministic frames, i.e. frames  $\mathcal{F} = (W, R, \star)$  such that for all  $x, y \in W$ ,  $card(x\star y) \leq 1$ ;
- serial frames, i.e. frames  $\mathcal{F} = (W, R, \star)$  such that for all  $x, y \in W$ ,  $card(x\star y) \geq 1$ .

In separated frames, there is at most one way to decompose a given state; in rich frames, there is at least one way to decompose a given state; in deterministic frames, there is at most one way to combine 2 given states; in serial frames, there is at least one way to combine 2 given states. Each frame considered in [5] is separated and deterministic whereas each frame considered in [13, Chapter 1] is separated, deterministic and serial.

EXAMPLE: Let  $W_1$  be the set of all words on an alphabet and  $\star_1$  be the operation of concatenation. The structure  $\mathcal{F}_1 = (W_1, \star_1)$  is not separated. Nevertheless, it is rich, deterministic and serial. Let  $W_2$  be the set of all binary trees and  $\star_2$  be the operation of join. The structure  $\mathcal{F}_2 = (W_2, \star_2)$  is not rich. Nevertheless, it is separated, deterministic and serial. Let  $W_3$  be the set of all heaps (partially defined functions mapping locations to values) and  $\star_3$  be the operation of union (undefined when

domains overlap). The structure  $\mathcal{F}_3 = (W_3, \star_3)$  is neither separated, nor serial. Nevertheless, it is rich and deterministic.

A model on  $\mathcal{F}$  is a 4-tuple  $\mathcal{M} = (W, R, \star, V)$  where  $V: p \mapsto V(p) \subseteq W$  is a valuation on  $\mathcal{F}$ , i.e. a function from the set of all atomic formulas into the set of all sets of states. In  $\mathcal{M}$ , programs are interpreted as binary relations over  $W$  and formulas are interpreted as subsets of  $W$  as follows:

- $(a)^\mathcal{M} = R(a)$ ;
- $(\phi?)^\mathcal{M} = \{(x, y): x = y \text{ and } y \in (\phi)^\mathcal{M}\}$ ;
- $(s_1)^\mathcal{M} = \{(x, y): \text{there exists } z \in W \text{ such that } y \in x \star z\}$ ;
- $(s_2)^\mathcal{M} = \{(x, y): \text{there exists } z \in W \text{ such that } y \in z \star x\}$ ;
- $(r_1)^\mathcal{M} = \{(x, y): \text{there exists } z \in W \text{ such that } x \in y \star z\}$ ;
- $(r_2)^\mathcal{M} = \{(x, y): \text{there exists } z \in W \text{ such that } x \in z \star y\}$ ;
- $(\alpha; \beta)^\mathcal{M} = \{(x, y): \text{there exists } z \in W \text{ such that } x(\alpha)^\mathcal{M} z \text{ and } z(\beta)^\mathcal{M} y\}$ ;
- $(\alpha \parallel \beta)^\mathcal{M} = \{(x, y): \text{there exists } z, t, u, v \in W \text{ such that } x \in z \star t, y \in u \star v, z(\alpha)^\mathcal{M} u \text{ and } t(\beta)^\mathcal{M} v\}$ ;
- $(p)^\mathcal{M} = V(p)$ ;
- $(\perp)^\mathcal{M}$  is empty;
- $(\neg\phi)^\mathcal{M} = W \setminus (\phi)^\mathcal{M}$ ;
- $(\phi \vee \psi)^\mathcal{M} = (\phi)^\mathcal{M} \cup (\psi)^\mathcal{M}$ ;
- $([\alpha]\phi)^\mathcal{M} = \{x: \text{for all } y \in W, \text{ if } x(\alpha)^\mathcal{M} y, y \in (\alpha)^\mathcal{M}\}$ .

The definition of the binary relation over  $W$  interpreting programs of the form  $\alpha \parallel \beta$  is in accordance with the definition given by Benevides *et al.* [5]. It says that to execute such a program from  $x$  to  $y$  consists in three steps: (i) decompose  $x$  into  $z$  and  $t$ ; (ii) from  $z$  and  $t$ , separately execute  $\alpha$  and  $\beta$  in parallel, thus reaching  $u, v$ ; (iii) combine  $u$  and  $v$  into  $y$ . Of course,  $\alpha \parallel \beta$  cannot be executed from  $x$  to  $y$  when it is not possible to decompose  $x$  and  $y$  in pairs  $(z, t), (u, v)$  such that  $z(\alpha)^\mathcal{M} u$  and  $t(\beta)^\mathcal{M} v$ . Obviously,

#### PROPOSITION 1

Let  $\mathcal{M} = (W, R, \star, V)$  be a model. For all programs  $\alpha$  and for all formulas  $\phi$ ,  $(\langle\alpha\rangle\phi)^\mathcal{M} = \{x: \text{there exists } y \in W \text{ such that } x(\alpha)^\mathcal{M} y \text{ and } y \in (\phi)^\mathcal{M}\}$ .

PROOF. By definition. Left to the reader. ■

We shall say that a formula  $\phi$  is valid in a model  $\mathcal{M} = (W, R, \star, V)$ , in symbols  $\mathcal{M} \models \phi$ , iff  $(\phi)^\mathcal{M} = W$ . A formula  $\phi$  is said to be valid in a frame  $\mathcal{F}$ , in symbols  $\mathcal{F} \models \phi$ , iff for all models  $\mathcal{M}$  on  $\mathcal{F}$ ,  $\mathcal{M} \models \phi$ . The validity in a frame  $\mathcal{F}$  of a set  $\Sigma$  of formulas, in symbols  $\mathcal{F} \models \Sigma$ , is defined in a similar way. We shall say that a formula  $\phi$  is valid in a class  $\mathcal{C}$  of frames, in symbols  $\mathcal{C} \models \phi$ , iff for all frames  $\mathcal{F}$  in  $\mathcal{C}$ ,  $\mathcal{F} \models \phi$ . A class  $\mathcal{C}$  of frames is said to be modally defined by a set  $\Sigma$  of formulas iff for all frames  $\mathcal{F}$ ,  $\mathcal{F}$  is in  $\mathcal{C}$  iff  $\mathcal{F} \models \Sigma$ . We shall say that a class of frames is modally definable iff it is modally defined by a set of formulas.

EXAMPLE: The class of all functional frames is modally defined by the formulas  $\langle a \rangle p \rightarrow [a]p$  for every atomic program  $a$ .

A model is said to be functional (respectively, separated, rich, deterministic, serial) iff it is based on a functional (respectively, separated, rich, deterministic, serial) frame. Let  $\mathcal{M} = (W, R, \star, V)$  be a model. The property ‘state  $z$  can be reached from state  $x$  by performing parametrized action  $\check{\alpha}$  via state  $y$  in  $\mathcal{M}$ ’—in symbols  $xR_\mathcal{M}(\check{\alpha}, y)z$ —and the property ‘admissible form  $\check{\phi}$  is true at state  $x$  via

state  $y$  in  $\mathcal{M}'$ —in symbols  $x \in V_{\mathcal{M}}(\check{\phi}, y)$ —are inductively defined as follows:

- $xR_{\mathcal{M}}(\neg\check{\phi}?, y)z$  iff  $x = z$  and  $z \in V_{\mathcal{M}}(\check{\phi}, y)$ ;
- $xR_{\mathcal{M}}(\check{\alpha}; \beta, y)z$  iff there exists  $t \in W$  such that  $xR_{\mathcal{M}}(\check{\alpha}, y)t$  and  $t(\beta)^{\mathcal{M}}z$ ;
- $xR_{\mathcal{M}}(\alpha; \check{\beta}, y)z$  iff there exists  $t \in W$  such that  $x(\alpha)^{\mathcal{M}}t$  and  $tR_{\mathcal{M}}(\check{\beta}, y)z$ ;
- $xR_{\mathcal{M}}(\check{\alpha} \parallel \check{\beta}, y)z$  iff there exists  $t, u, v, w \in W$  such that  $x \in t \star u$ ,  $z \in v \star w$ ,  $tR_{\mathcal{M}}(\check{\alpha}, y)v$  and  $u(\beta)^{\mathcal{M}}w$ ;
- $xR_{\mathcal{M}}(\alpha \parallel \check{\beta}, y)z$  iff there exists  $t, u, v, w \in W$  such that  $x \in t \star u$ ,  $z \in v \star w$ ,  $t(\alpha)^{\mathcal{M}}v$  and  $uR_{\mathcal{M}}(\check{\beta}, y)w$ ;
- $x \in V_{\mathcal{M}}(\sharp, y)$  iff  $x = y$ ;
- $x \in V_{\mathcal{M}}([\check{\alpha}] \perp, y)$  iff there exists  $z \in W$  such that  $xR_{\mathcal{M}}(\check{\alpha}, y)z$ .

These properties are quite abstract. The following Proposition can help the reader to grasp what they mean.

**PROPOSITION 2**

Let  $\mathcal{M} = (W, R, \star, V)$  be a model. For all expressions  $e\check{x}p$  (either a parametrized action, or an admissible form),

- if  $e\check{x}p$  is a parametrized action, for all formulas  $\phi$  and for all  $x, z \in W$ ,  $x(e\check{x}p(\phi))^{\mathcal{M}}z$  iff there exists  $y \in W$  such that  $xR_{\mathcal{M}}(e\check{x}p, y)z$  and  $y \notin (\phi)^{\mathcal{M}}$ ;
- if  $e\check{x}p$  is an admissible form, for all formulas  $\phi$  and for all  $x \in W$ ,  $x \in (e\check{x}p(\phi))^{\mathcal{M}}$  iff for all  $y \in W$ , if  $x \in V_{\mathcal{M}}(e\check{x}p, y)$ ,  $y \in (\phi)^{\mathcal{M}}$ .

**PROOF.** By induction on  $e\check{x}p$ . Left to the reader. ■

We will make use of Proposition 2 when we establish the soundness for  $PRSPDL_0$ .

Let  $\mathcal{M} = (W, R, \star, V)$ ,  $\mathcal{M}' = (W', R', \star', V')$  be models and  $p$  be an atomic formula. We shall say that  $\mathcal{M}$  and  $\mathcal{M}'$  are  $p$ -similar, in symbols  $\mathcal{M} \sim_p \mathcal{M}'$ , if  $W = W'$ ,  $R = R'$ ,  $\star = \star'$  and for all atomic formulas  $q$ , if  $p \neq q$ ,  $V(q) = V'(q)$ . When  $\mathcal{M} \sim_p \mathcal{M}'$ , we will also write  $V \sim_p V'$ . Obviously,

**PROPOSITION 3**

Let  $\mathcal{M} = (W, R, \star, V)$ ,  $\mathcal{M}' = (W', R', \star', V')$  be models and  $p$  be an atomic formula. If  $\mathcal{M} \sim_p \mathcal{M}'$ ,

- for all expressions  $exp$  (either a program, or a formula), if  $p$  does not occur in  $exp$ ,  $(exp)^{\mathcal{M}} = (exp)^{\mathcal{M}'}$ ;
- for all parametrized actions  $\check{\alpha}$ , if  $p$  does not occur in  $\check{\alpha}$ ,  $R_{\mathcal{M}}(\check{\alpha}, \cdot) = R_{\mathcal{M}'}(\check{\alpha}, \cdot)$ ;
- for all admissible forms  $\check{\phi}$ , if  $p$  does not occur in  $\check{\phi}$ ,  $V_{\mathcal{M}}(\check{\phi}, \cdot) = V_{\mathcal{M}'}(\check{\phi}, \cdot)$ .

**PROOF.** By induction on  $exp$ ,  $\check{\alpha}$  and  $\check{\phi}$ . Left to the reader. ■

The next four Propositions present valid formulas and rules of proof preserving validity.

**PROPOSITION 4 (Validity 1)**

The following formulas are valid in the class of all frames:

- (A1) all instances of propositional tautologies;
- (A2)  $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$ ;
- (A3)  $[\phi?]\psi \leftrightarrow (\phi \rightarrow \psi)$ ;
- (A4)  $\phi \rightarrow [s_1]\langle r_1 \rangle \phi$ ;
- (A5)  $\phi \rightarrow [s_2]\langle r_2 \rangle \phi$ ;
- (A6)  $\phi \rightarrow [r_1]\langle s_1 \rangle \phi$ ;
- (A7)  $\phi \rightarrow [r_2]\langle s_2 \rangle \phi$ ;
- (A8)  $\langle r_1 \rangle \top \leftrightarrow \langle r_2 \rangle \top$ ;
- (A9)  $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$ ;



- (A10)  $\langle \alpha \parallel \beta \rangle \phi \rightarrow \langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge \psi) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg \psi)$ ;  
 (A11)  $\langle \alpha(\phi?) \rangle \top \rightarrow \langle \alpha((\phi \wedge \psi)?) \rangle \top \vee \langle \alpha((\phi \wedge \neg \psi)?) \rangle \top$ ;  
 (A12)  $\langle \alpha \rangle \phi \leftrightarrow \langle f(\alpha) \rangle \phi$ .

PROOF. For (A1)–(A10), by definition. For (A11) and (A12), by induction on  $\alpha$ . Left to the reader. ■

PROPOSITION 5 (Validity 2)

The following formulas are valid in the class of all separated frames:

- (A13)  $\langle r_1 \rangle \phi \rightarrow [r_1] \phi$ ;  
 (A14)  $\langle r_2 \rangle \phi \rightarrow [r_2] \phi$ .

PROOF. By definition. Left to the reader. ■

PROPOSITION 6 (Admissibility 1)

The following rules of proof preserve validity in the class of all frames:

- (MP) from  $\phi, \phi \rightarrow \psi$ , infer  $\psi$ ;  
 (N) from  $\phi$ , infer  $[\alpha] \phi$ .

PROOF. The rules of proof (MP) and (N) are probably familiar to the reader. See [7, Chapter 1] for the proof that they preserve validity in the class of all separated frames. ■

PROPOSITION 7 (Admissibility 2)

The following rule of proof preserve validity in the class of all separated frames:

- (FOR) from  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$ :  $p$  is an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$ , infer  $\check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$ .

PROOF. Suppose (FOR) does not preserve validity in the class of all separated frames. Hence, there exists an admissible form  $\check{\phi}$ , there exists programs  $\alpha, \beta$  and there exists a formula  $\psi$  such that for all atomic formulas  $p$  not occurring in  $\check{\phi}, \alpha, \beta, \psi$ ,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  is valid in the class of all separated frames and  $\check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$  is not valid in the class of all separated frames. Thus, there exists a separated frame  $\mathcal{F} = (W, R, \star)$  such that  $\mathcal{F} \not\models \check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$ . Therefore, there exists a model  $\mathcal{M} = (W, R, \star, V)$  on  $\mathcal{F}$  such that  $\mathcal{M} \not\models \check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$ . Consequently,  $(\check{\phi}(\langle \alpha \parallel \beta \rangle \psi))^{\mathcal{M}} \neq W$ . Hence, there exists  $x \in W$  such that  $x \notin (\check{\phi}(\langle \alpha \parallel \beta \rangle \psi))^{\mathcal{M}}$ . By Proposition 2, there exists  $y \in W$  such that  $x \in V_{\mathcal{M}}(\check{\phi}, y)$  and  $y \notin (\langle \alpha \parallel \beta \rangle \psi)^{\mathcal{M}}$ . Let  $p$  be an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$  and  $V'$ :  $q \mapsto V'(q) \subseteq W$  be a valuation on  $\mathcal{F}$  such that  $V' \sim_p V$  and  $V'(p) = \{z : \text{there exists } t, u, v, w \in W \text{ such that } y \in t \star u, z \in v \star w \text{ and } u(\beta)^{\mathcal{M}}_w\}$ . By Proposition 3, since  $x \in V_{\mathcal{M}}(\check{\phi}, y)$ ,  $x \in V_{(W, R, \star, V')}(\check{\phi}, y)$ . Since for all atomic formulas  $p$  not occurring in  $\check{\phi}, \alpha, \beta, \psi$ ,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  is valid in the class of all separated frames and  $\mathcal{F}$  is separated,  $\mathcal{F} \models \check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$ . Thus,  $(W, R, \star, V') \models \check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$ . Therefore,  $(\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p)))^{(W, R, \star, V')} = W$ . Consequently,  $x \in (\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p)))^{(W, R, \star, V')}$ . By Proposition 2, since  $x \in V_{(W, R, \star, V')}(\check{\phi}, y)$ ,  $y \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))^{(W, R, \star, V')}$ . Hence, either  $y \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p))^{(W, R, \star, V')}$ , or  $y \in (\langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))^{(W, R, \star, V')}$ .

**Case** ‘ $y \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p))^{(W, R, \star, V')}$ ’. Hence, there exists  $z, t, u, v, w \in W$  such that  $y \in t \star u$ ,  $z \in v \star w$ ,  $t(\alpha)^{(W, R, \star, V')}_v$ ,  $z \in (\psi)^{(W, R, \star, V')}$  and  $z \in V'(p)$ . By Proposition 3, since  $p$  does not occur in  $\alpha, \psi$  and  $V' \sim_p V$ ,  $t(\alpha)^{\mathcal{M}}_v$  and  $z \in (\psi)^{\mathcal{M}}$ . Since  $\mathcal{M}$  is separated,  $y \in t \star u$ ,  $z \in v \star w$  and  $z \in V'(p)$ ,  $u(\beta)^{\mathcal{M}}_w$ . Since  $y \in t \star u$ ,  $z \in v \star w$  and  $t(\alpha)^{\mathcal{M}}_v$ ,  $y(\alpha \parallel \beta)^{\mathcal{M}}_z$ . Since  $z \in (\psi)^{\mathcal{M}}$ ,  $y \in (\langle \alpha \parallel \beta \rangle \psi)^{\mathcal{M}}$ : a contradiction.



**Case** ‘ $y \in (\langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))^{(W, R, \star, V')}$ ’. Hence, there exists  $z, t, u, v, w \in W$  such that  $y \in t \star u$ ,  $z \in v \star w$ ,  $u(\beta)^{(W, R, \star, V')}_w$ ,  $z \in (\psi)^{(W, R, \star, V')}$  and  $z \notin V'(p)$ . By Proposition 3, since  $p$  does not occur in  $\beta$  and  $V' \sim_p V$ ,  $u(\beta)^{\mathcal{M}}_w$ . Since  $y \in t \star u$  and  $z \in v \star w$ ,  $z \in V'(p)$ : a contradiction. ■

The axiomatization of  $PRSPDL_0$  given in Section 6 will be based on the formulas and the rules of proof contained in Propositions 4–7.

## 4 Expressivity

About expressivity, we now illustrate the interest of our new constructs for programs and formulas. More precisely, we show that, in the class of all separated models, the following constructs for programs cannot be eliminated without strictly weakening the expressivity of the language: tests (Proposition 8), storing programs (Proposition 9), recovering programs (Proposition 10), sequential composition (Proposition 11) and parallel composition (Proposition 12).

### PROPOSITION 8

For all test-free formulas  $\phi$ , the formulas  $\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top$  and  $\phi$  are not equally interpreted in all separated models.

**PROOF.** Suppose there exists a test-free formula  $\phi$  from the language of  $PRSPDL_0$  such that the formulas  $\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top$  and  $\phi$  are equally interpreted in all separated models. Without loss of generality, assume  $a, b$  are the only program variables in  $\phi$  and  $\phi$  contains no propositional variable. Moreover, in this proof, we will assume that  $a$  and  $b$  are the only syntactic elements occurring in programs and formulas. Let  $\mathcal{F} = (W, R, \star)$  and  $\mathcal{F}' = (W', R', \star')$  be the separated frames defined as follows:  $W = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$ ,  $R(a) = \{(x_2, x_5), (x_3, x_4), (x_3, x_9), (x_4, x_6), (x_9, x_{11})\}$ ,  $R(b) = \{(x_4, x_8)\}$  and  $R$  is otherwise empty,  $x_2 \star x_3 = \{x_1\}$ ,  $x_5 \star x_6 = \{x_7\}$ ,  $x_{10} \star x_{11} = \{x_{12}\}$  and  $\star$  is otherwise empty,  $W' = \{x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8, x'_9, x'_{10}, x'_{11}, x'_{12}\}$ ,  $R'(a) = \{(x'_2, x'_5), (x'_3, x'_4), (x'_3, x'_9), (x'_4, x'_{11}), (x'_9, x'_6)\}$ ,  $R'(b) = \{(x'_4, x'_8)\}$  and  $R'$  is otherwise empty,  $x'_2 \star' x'_3 = \{x'_1\}$ ,  $x'_5 \star' x'_6 = \{x'_7\}$ ,  $x'_{10} \star' x'_{11} = \{x'_{12}\}$  and  $\star'$  is otherwise empty. Since  $\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames,  $\mathcal{F} \models \langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi$ . Let us consider the following binary relation:

- $Z = \{(x_1, x'_1), (x_2, x'_2), (x_3, x'_3), (x_4, x'_4), (x_5, x'_5), (x_5, x'_{10}), (x_6, x'_6), (x_6, x'_{11}), (x_7, x'_7), (x_7, x'_{12}), (x_8, x'_8), (x_9, x'_9), (x_{10}, x'_5), (x_{10}, x'_{10}), (x_{11}, x'_6), (x_{11}, x'_{11}), (x_{12}, x'_7), (x_{12}, x'_{12})\}$ .

Let  $\mathcal{M} = (W, R, \star, V)$  be a model on  $\mathcal{F}$  and  $\mathcal{M}' = (W', R', \star', V')$  be a model on  $\mathcal{F}'$ . Obviously,  $x_1 \in (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top)^{\mathcal{M}}$  and  $x'_1 \notin (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top)^{\mathcal{M}'}$ . Since  $\mathcal{F} \models \langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi$ ,  $x_1 \in (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi)^{\mathcal{M}}$  and  $x'_1 \in (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top \leftrightarrow \phi)^{\mathcal{M}'}$ . Since  $x_1 \in (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top)^{\mathcal{M}}$  and  $x'_1 \notin (\langle a \parallel (a; \langle b \rangle \top?; a) \rangle \top)^{\mathcal{M}'}$ ,  $x_1 \in (\phi)^{\mathcal{M}}$  and  $x'_1 \notin (\phi)^{\mathcal{M}'}$ .

**CLAIM:** Let  $\alpha$  be a test-free program from the language of  $PRSPDL_0$ . For all  $u \in W$  and for all  $u' \in W'$ ,

- if  $u(\alpha)^{\mathcal{M}}_{x_9}$ ,  $u(\alpha)^{\mathcal{M}}_{x_4}$ ;
- if  $u'(\alpha)^{\mathcal{M}'}_{x'_4}$ ,  $u'(\alpha)^{\mathcal{M}'}_{x'_9}$ .

**PROOF:** By induction on  $\alpha$ . Left to the reader.

**CLAIM:** Let  $\alpha$  be a test-free program from the language of  $PRSPDL_0$ . For all  $u, v \in W$  and for all  $u', v' \in W'$ ,

- if  $u \in \{x_5, x_6, x_7\}$  and  $u(\alpha)^{\mathcal{M}}v$ ,  $v \in \{x_5, x_6, x_7\}$ ;
- if  $u' \in \{x'_5, x'_6, x'_7\}$  and  $u'(\alpha)^{\mathcal{M}'}v'$ ,  $v' \in \{x'_5, x'_6, x'_7\}$ ;
- if  $u \in \{x_{10}, x_{11}, x_{12}\}$  and  $u(\alpha)^{\mathcal{M}}v$ ,  $v \in \{x_{10}, x_{11}, x_{12}\}$ ;
- if  $u' \in \{x'_{10}, x'_{11}, x'_{12}\}$  and  $u'(\alpha)^{\mathcal{M}'}v'$ ,  $v' \in \{x'_{10}, x'_{11}, x'_{12}\}$ .

PROOF: By induction on  $\alpha$ . Left to the reader.

CLAIM: Let  $\alpha$  be a test-free program from the language of  $PRSPDL_0$ . For all  $u \in W \setminus \{x_9\}$  and for all  $u' \in W' \setminus \{x'_4\}$ ,

- if  $u(\alpha)^{\mathcal{M}}x_{10}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $uZw'$ ,  $vZw'$  and  $v(\alpha)^{\mathcal{M}}x_5$ ;
- if  $u'(\alpha)^{\mathcal{M}'}x'_{10}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $vZu'$ ,  $vZw'$  and  $w'(\alpha)^{\mathcal{M}'}x'_5$ ;
- if  $u(\alpha)^{\mathcal{M}}x_{11}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $uZw'$ ,  $vZw'$  and  $v(\alpha)^{\mathcal{M}}x_6$ ;
- if  $u'(\alpha)^{\mathcal{M}'}x'_{11}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $vZu'$ ,  $vZw'$  and  $w'(\alpha)^{\mathcal{M}'}x'_6$ ;
- if  $u(\alpha)^{\mathcal{M}}x_{12}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $uZw'$ ,  $vZw'$  and  $v(\alpha)^{\mathcal{M}}x_7$ ;
- if  $u'(\alpha)^{\mathcal{M}'}x'_{12}$ , there exists  $v \in W$  and there exists  $w' \in W'$  such that  $vZu'$ ,  $vZw'$  and  $w'(\alpha)^{\mathcal{M}'}x'_7$ .

PROOF: By induction on  $\alpha$ . Left to the reader.

CLAIM: Let  $\alpha$  be a test-free program and  $\psi$  be a test-free formula from the language of  $PRSPDL_0$ . For all  $r \in W$  and for all  $r' \in W'$ , if  $rZr'$ ,

- for all  $s \in W$ , if  $r(\alpha)^{\mathcal{M}}s$ , there exists  $s' \in W'$  such that  $r'(\alpha)^{\mathcal{M}'}s'$  and  $sZs'$ ;
- for all  $s' \in W'$ , if  $r'(\alpha)^{\mathcal{M}'}s'$ , there exists  $s \in W$  such that  $r(\alpha)^{\mathcal{M}}s$  and  $sZs'$ ;
- $r \in (\psi)^{\mathcal{M}}$  iff  $r' \in (\psi)^{\mathcal{M}'}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader.

Since  $\phi$  is test-free,  $x_1Zx'_1$  and  $x_1 \in (\phi)^{\mathcal{M}}$ ,  $x'_1 \in (\phi)^{\mathcal{M}'}$ : a contradiction. ■

PROPOSITION 9

- For all  $s_1$ -free formulas  $\phi$ , the formulas  $\langle s_1 \rangle \top$  and  $\phi$  are not equally interpreted in all separated models;
- for all  $s_2$ -free formulas  $\phi$ , the formulas  $\langle s_2 \rangle \top$  and  $\phi$  are not equally interpreted in all separated models.

PROOF. Suppose there exists an  $s_1$ -free formula  $\phi$  from the language of  $PRSPDL_0$  such that  $\langle s_1 \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames. Without loss of generality, assume  $\phi$  contains neither program variable, nor propositional variable. Moreover, in this proof, we will assume that programs and formulas contain no syntactic element. Let  $\mathcal{F} = (W, R, \star)$  and  $\mathcal{F}' = (W', R', \star')$  be the separated frames defined as follows:  $W = \{x, y\}$ ,  $R$  is the empty function,  $x \star x = \{y\}$  and otherwise  $\star$  is the empty function,  $W' = \{x', y'\}$ ,  $R'$  is the empty function and  $\star'$  is the empty function. Since  $\langle s_1 \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames,  $\mathcal{F} \models \langle s_1 \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle s_1 \rangle \top \leftrightarrow \phi$ . Let  $\mathcal{M} = (W, R, \star, V)$  be a model on  $\mathcal{F}$  and  $\mathcal{M}' = (W', R', \star', V')$  be a model on  $\mathcal{F}'$ . Obviously,  $x \in (\langle s_1 \rangle \top)^{\mathcal{M}}$  and  $x' \notin (\langle s_1 \rangle \top)^{\mathcal{M}'}$ . Since  $\mathcal{F} \models \langle s_1 \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle s_1 \rangle \top \leftrightarrow \phi$ ,  $x \in (\langle s_1 \rangle \top \leftrightarrow \phi)^{\mathcal{M}}$  and  $x' \in (\langle s_1 \rangle \top \leftrightarrow \phi)^{\mathcal{M}'}$ . Since  $x \in (\langle s_1 \rangle \top)^{\mathcal{M}}$  and  $x' \notin (\langle s_1 \rangle \top)^{\mathcal{M}'}$ ,  $x \in (\phi)^{\mathcal{M}}$  and  $x' \notin (\phi)^{\mathcal{M}'}$ .

CLAIM: Let  $\alpha$  be an  $s_1$ -free program from the language of  $PRSPDL_0$ . For all  $r \in W$  and for all  $r' \in W'$ ,

- if  $x(\alpha)^{\mathcal{M}}r$ ,  $r = x$ ;
- if  $x'(\alpha)^{\mathcal{M}'}r'$ ,  $r' = x'$ .

PROOF: By induction on  $\alpha$ . Left to the reader.

CLAIM: Let  $\alpha$  be an  $s_1$ -free program and  $\psi$  be an  $s_1$ -free formula from the language of  $PRSPDL_0$ . Then,

- $x(\alpha)^{\mathcal{M}}x$  iff  $x'(\alpha)^{\mathcal{M}'}x'$ ;
- $x \in (\psi)^{\mathcal{M}}$  iff  $x' \in (\psi)^{\mathcal{M}'}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader.

Since  $\phi$  is  $s_1$ -free and  $x \in (\phi)^{\mathcal{M}}$ ,  $x' \in (\phi)^{\mathcal{M}'}$ : a contradiction.

The argument concerning  $s_2$  is similar to the previous argument. ■

PROPOSITION 10

- for all  $r_1$ -free formulas  $\phi$ , the formulas  $\langle r_1 \parallel \top? \rangle \top$  and  $\phi$  are not equally interpreted in all separated models;
- for all  $r_2$ -free formulas  $\phi$ , the formulas  $\langle \top? \parallel r_2 \rangle \top$  and  $\phi$  are not equally interpreted in all separated models.

PROOF. Suppose there exists an  $r_1$ -free formula  $\phi$  from the language of  $PRSPDL_0$  such that the formulas  $\langle r_1 \parallel \top? \rangle \top$  and  $\phi$  are equally interpreted in all separated models. Without loss of generality, assume  $\phi$  contains neither program variable, nor propositional variable. Moreover, in this proof, we will assume that programs and formulas contain no syntactic element. Let  $\mathcal{F} = (W, R, \star)$  be the separated frame defined as follows:  $W = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$ ,  $R$  is the empty function,  $x_4 \star x_5 = \{x_1\}$ ,  $x_6 \star x_5 = \{x_2\}$ ,  $x_6 \star x_7 = \{x_3\}$ ,  $x_8 \star x_9 = \{x_4\}$ ,  $x_{10} \star x_{11} = \{x_6\}$ ,  $x_8 \star x_5 = \{x_{12}\}$ ,  $x_{10} \star x_7 = \{x_{13}\}$  and  $\star$  is otherwise empty. Since  $\langle r_1 \parallel \top? \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames,  $\mathcal{F} \models \langle r_1 \parallel \top? \rangle \top \leftrightarrow \phi$ . Let us consider the following partition of  $W$ :

- $W/Z = \{\{x_1, x_2, x_3\}, \{x_4, x_6\}, \{x_5, x_7\}, \{x_8, x_{10}\}, \{x_9, x_{11}\}, \{x_{12}, x_{13}\}\}$ .

Let  $\mathcal{M} = (W, R, \star, V)$  be a model on  $\mathcal{F}$ . Obviously,  $x_1 \in (\langle r_1 \parallel \top? \rangle \top)^{\mathcal{M}}$  and  $x_2 \notin (\langle r_1 \parallel \top? \rangle \top)^{\mathcal{M}}$ . Since  $\mathcal{F} \models \langle r_1 \parallel \top? \rangle \top \leftrightarrow \phi$ ,  $x_1 \in (\langle r_1 \parallel \top? \rangle \top \leftrightarrow \phi)^{\mathcal{M}}$  and  $x_2 \in (\langle r_1 \parallel \top? \rangle \top \leftrightarrow \phi)^{\mathcal{M}}$ . Since  $x_1 \in (\langle r_1 \parallel \top? \rangle \top)^{\mathcal{M}}$  and  $x_2 \notin (\langle r_1 \parallel \top? \rangle \top)^{\mathcal{M}}$ ,  $x_1 \in (\phi)^{\mathcal{M}}$  and  $x_2 \notin (\phi)^{\mathcal{M}}$ . Let  $W_L = \{x_1, x_2, x_4, x_5, x_8, x_9, x_{12}\}$  and  $W_R = \{x_3, x_6, x_7, x_{10}, x_{11}, x_{13}\}$ . Let  $f_{LR}$  be the function from  $L$  into  $R$  inductively defined as follows:  $f_{LR}(1)=3, f_{LR}(2)=3, f_{LR}(4)=6, f_{LR}(5)=7, f_{LR}(8)=10, f_{LR}(9)=11$  and  $f_{LR}(12)=13$ . Let  $f_{RL}$  be the function from  $R$  into  $L$  inductively defined as follows:  $f_{RL}(3)=1, f_{RL}(6)=4, f_{RL}(7)=5, f_{RL}(10)=8, f_{RL}(11)=9$  and  $f_{RL}(13)=12$ .

CLAIM: Let  $\alpha$  be an  $r_1$ -free program from the language of  $PRSPDL_0$ . For all  $u, v \in W$ , if  $u(\alpha)^{\mathcal{M}}v$ ,

- if  $u \in W_L, v \in W_L$ ;
- if  $v \in \{x_4, x_8, x_9\}, u \in \{x_4, x_8, x_9\}$ ;
- if  $v = x_8, u = x_8$ ;
- if  $v \in \{x_6, x_{10}, x_{11}\}, u \in \{x_6, x_{10}, x_{11}\}$ ;
- if  $v = x_{10}, u = x_{10}$ .

PROOF: By induction on  $\alpha$ . Left to the reader.

CLAIM: Let  $\alpha$  be an  $r_1$ -free program and  $\psi$  be an  $r_1$ -free formula from the language of  $PRSPDL_0$ . For all  $u, v \in W$ , if  $u(\alpha)^{\mathcal{M}}v$ ,

- if  $u, v \in W_L, f(u)(\alpha)^{\mathcal{M}}f(v)$ ;
- if  $u, v \in W_R, g(u)(\alpha)^{\mathcal{M}}g(v)$ ;
- if  $u \in W_R$  and  $v \in W_L, u(\alpha)^{\mathcal{M}}f(v)$ .

For all  $u, v \in W$ , if  $uZv$ ,

- for all  $s \in W$ , if  $u(\alpha)^{\mathcal{M}}s$ , there exists  $t \in W$  such that  $v(\alpha)^{\mathcal{M}}t$  and  $sZt$ ;
- for all  $t \in W$ , if  $v(\alpha)^{\mathcal{M}}t$ , there exists  $s \in W$  such that  $u(\alpha)^{\mathcal{M}}s$  and  $sZt$ ;
- $u \in (\psi)^{\mathcal{M}}$  iff  $v \in (\psi)^{\mathcal{M}}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader.

Since  $\phi$  is  $r_1$ -free,  $x_1Zx_2$  and  $x_1 \in (\phi)^{\mathcal{M}}$ ,  $x_2 \in (\phi)^{\mathcal{M}}$ : a contradiction.

The argument concerning  $r_2$  is similar to the previous argument. ■

#### PROPOSITION 11

Let  $a$  be an atomic program. For all  $\phi$  ; -free formulas  $\phi$ , the formulas  $\langle (a; a) \parallel a \rangle \top$  and  $\phi$  are not equally interpreted in all separated models.

PROOF. Suppose there exists a ; -free formula  $\phi$  from the language of  $PRSPDL_0$  such that the formulas  $\langle (a; a) \parallel a \rangle \top$  and  $\phi$  are equally interpreted in all separated models. Without loss of generality, assume  $a$  is the only program variable in  $\phi$  and  $\phi$  contains no propositional variable. Moreover, in this proof, we will assume that  $a$  is the only syntactic element occurring in programs and formulas. Let  $\mathcal{F} = (W, R, \star)$  and  $\mathcal{F}' = (W', R', \star')$  be the separated frames defined as follows:  $W = \{x, y, z, t, u, v, w\}$ ,  $R(a) = \{(y, w), (w, t), (z, u)\}$  and  $R$  is otherwise empty,  $y \star z = \{x\}$ ,  $t \star u = \{v\}$  and  $\star$  is otherwise empty,  $W' = \{x', y', z', t'_1, t'_2, u'_1, u'_2, v'_1, v'_2, w'\}$ ,  $R'(a) = \{(y', w'), (w', t'_1), (z', u'_2)\}$  and  $R'$  is otherwise empty and  $y' \star' z' = \{x'\}$ ,  $t'_1 \star' u'_1 = \{v'_1\}$ ,  $t'_2 \star' u'_2 = \{v'_2\}$  and  $\star'$  is otherwise empty. Since  $\langle (a; a) \parallel a \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames,  $\mathcal{F} \models \langle (a; a) \parallel a \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle (a; a) \parallel a \rangle \top \leftrightarrow \phi$ . Let us consider the following binary relation:

- $Z = \{(x, x'), (y, y'), (z, z'), (t, t'_1), (t, t'_2), (u, u'_1), (u, u'_2), (v, v'_1), (v, v'_2), (w, w')\}$ .

Let  $\mathcal{M} = (W, R, *, V)$  be a model on  $\mathcal{F}$  and  $\mathcal{M}' = (W', R', *', V')$  be a model on  $\mathcal{F}'$ . Obviously,  $x \in ((a; a) \parallel a) \top^{\mathcal{M}}$  and  $x' \notin ((a; a) \parallel a) \top^{\mathcal{M}'}$ . Since  $\mathcal{F} \models \langle (a; a) \parallel a \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle (a; a) \parallel a \rangle \top \leftrightarrow \phi$ ,  $x \in ((a; a) \parallel a) \top \leftrightarrow \phi^{\mathcal{M}}$  and  $x' \in ((a; a) \parallel a) \top \leftrightarrow \phi^{\mathcal{M}'}$ . Since  $x \in ((a; a) \parallel a) \top^{\mathcal{M}}$  and  $x' \notin ((a; a) \parallel a) \top^{\mathcal{M}'}$ ,  $x \in (\phi)^{\mathcal{M}}$  and  $x' \notin (\phi)^{\mathcal{M}'}$ .

CLAIM: Let  $\alpha$  be a ; -free program from the language of  $PRSPDL_0$ . For all  $r \in W$  and for all  $r' \in W'$ ,

- if  $y(\alpha)^{\mathcal{M}}r$ ,  $r \in \{x, y, w\}$ ;
- if  $y'(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{x', y', w'\}$ ;
- if  $z(\alpha)^{\mathcal{M}}r$ ,  $r \in \{x, z, u\}$ ;
- if  $z'(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{x', z', u'\}$ ;
- if  $t(\alpha)^{\mathcal{M}}r$ ,  $r \in \{t, v\}$ ;
- if  $t'_1(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{t'_1, v'_1\}$ ;
- if  $t'_2(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{t'_2, v'_2\}$ ;
- if  $u(\alpha)^{\mathcal{M}}r$ ,  $r \in \{u, v\}$ ;
- if  $u'_1(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{u'_1, v'_1\}$ ;
- if  $u'_2(\alpha)^{\mathcal{M}'}r'$ ,  $r' \in \{u'_2, v'_2\}$ .

PROOF: By induction on  $\alpha$ . Left to the reader.

CLAIM: Let  $\alpha$  be a ; -free program and  $\psi$  be a ; -free formula from the language of  $PRSPDL_0$ . For all  $r \in W$  and for all  $r' \in W'$ , if  $rZr'$ ,

- for all  $s \in W$ , if  $r(\alpha)^{\mathcal{M}}s$ , there exists  $s' \in W'$  such that  $r'(\alpha)^{\mathcal{M}'}s'$  and  $sZs'$ ;
- for all  $s' \in W'$ , if  $r'(\alpha)^{\mathcal{M}'}s'$ , there exists  $s \in W$  such that  $r(\alpha)^{\mathcal{M}}s$  and  $sZs'$ ;
- $r \in (\psi)^{\mathcal{M}}$  iff  $r' \in (\psi)^{\mathcal{M}'}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader. Since  $\phi$  is ;-free,  $xZx'$  and  $x \in (\phi)^{\mathcal{M}}$ ,  $x' \in (\phi)^{\mathcal{M}'}$ : a contradiction. ■

#### PROPOSITION 12

Let  $a$  be an atomic program. For all  $\parallel$ -free formulas  $\phi$ , the formulas  $\langle a \parallel a \rangle \top$  and  $\phi$  are not equally interpreted in all separated models.

PROOF. Suppose there exists a  $\parallel$ -free formula  $\phi$  from the language of  $PRSPDL_0$  such that the formulas  $\langle a \parallel a \rangle \top$  and  $\phi$  are equally interpreted in all separated models. Without loss of generality, assume  $a$  is the only program variable in  $\phi$  and  $\phi$  contains no propositional variable. Moreover, in this proof, we will assume that  $a$  is the only syntactic element occurring in programs and formulas. Let  $\mathcal{F} = (W, R, \star)$  and  $\mathcal{F}' = (W', R', \star')$  be the separated frames defined as follows:  $W = \{x, y, z, t, u, v\}$ ,  $R(a) = \{(y, t), (z, u)\}$  and  $R$  is otherwise empty,  $y \star z = \{x\}$ ,  $t \star u = \{v\}$  and  $\star$  is otherwise empty,  $W' = \{x', y', z', t', u', v'\}$ ,  $R'(a) = \{(y', t'), (z', u')\}$  and  $R'$  is otherwise empty and  $y' \star' z' = \{x'\}$ ,  $t' \star' u' = \{v'\}$  and  $\star'$  is otherwise empty. Since  $\langle a \parallel a \rangle \top \leftrightarrow \phi$  is valid in the class of all separated frames,  $\mathcal{F} \models \langle a \parallel a \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle a \parallel a \rangle \top \leftrightarrow \phi$ . Let us consider the following binary relation:

- $Z = \{(x, x'), (y, y'), (z, z'), (t, t'), (t, t'), (u, u'), (u, u'), (v, v'), (v, v'), (w, w')\}$ .

Let  $\mathcal{M} = (W, R, \star, V)$  be a model on  $\mathcal{F}$  and  $\mathcal{M}' = (W', R', \star', V')$  be a model on  $\mathcal{F}'$ . Obviously,  $x \in (\langle a \parallel a \rangle \top)^{\mathcal{M}}$  and  $x' \notin (\langle a \parallel a \rangle \top)^{\mathcal{M}'}$ . Since  $\mathcal{F} \models \langle a \parallel a \rangle \top \leftrightarrow \phi$  and  $\mathcal{F}' \models \langle a \parallel a \rangle \top \leftrightarrow \phi$ ,  $x \in (\langle a \parallel a \rangle \top \leftrightarrow \phi)^{\mathcal{M}}$  and  $x' \in (\langle a \parallel a \rangle \top \leftrightarrow \phi)^{\mathcal{M}'}$ . Since  $x \in (\langle a \parallel a \rangle \top)^{\mathcal{M}}$  and  $x' \notin (\langle a \parallel a \rangle \top)^{\mathcal{M}'}$ ,  $x \in (\phi)^{\mathcal{M}}$  and  $x' \notin (\phi)^{\mathcal{M}'}$ .

CLAIM: Let  $\alpha$  be a  $\parallel$ -free program and  $\psi$  be a  $\parallel$ -free formula from the language of  $PRSPDL_0$ . For all  $r \in W$  and for all  $r' \in W'$ , if  $rZr'$ ,

- for all  $s \in W$ , if  $r(\alpha)^{\mathcal{M}}s$ , there exists  $s' \in W'$  such that  $r'(\alpha)^{\mathcal{M}'}s'$  and  $sZs'$ ;
- for all  $s' \in W'$ , if  $r'(\alpha)^{\mathcal{M}'}s'$ , there exists  $s \in W$  such that  $r(\alpha)^{\mathcal{M}}s$  and  $sZs'$ ;
- $r \in (\psi)^{\mathcal{M}}$  iff  $r' \in (\psi)^{\mathcal{M}'}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader.

Since  $\phi$  is  $\parallel$ -free,  $xZx'$  and  $x \in (\phi)^{\mathcal{M}}$ ,  $x' \in (\phi)^{\mathcal{M}'}$ : a contradiction. ■

It should be clear from Propositions 8–12 that neither tests, nor the storing programs  $s_1$  and  $s_2$  and the recovering programs  $r_1$  and  $r_2$ , nor the program construct  $(\cdot; \cdot)$  of sequential composition, nor the program construct  $(\cdot \parallel \cdot)$  of parallel composition can be defined in terms of the other constructs of the language of  $PRSPDL_0$ . Nevertheless,

#### PROPOSITION 13

Let  $\mathcal{M} = (W, R, \star, V)$  be a separated model and  $x \in W$ . For all programs  $\alpha, \beta$ , for all formulas  $\phi$  and for all atomic formulas  $p$ , if  $p$  does not occur in  $\alpha, \beta, \phi$ , the following conditions are equivalent:

- (1)  $x \in (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$ ;
- (2) for all separated models  $\mathcal{M}' = (W', R', \star', V')$ , if  $\mathcal{M} \sim_p \mathcal{M}'$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ .

PROOF. Suppose there exists programs  $\alpha, \beta$ , there exists a formula  $\phi$  and there exists an atomic formula  $p$  such that  $p$  does not occur in  $\alpha, \beta, \phi$  and the above conditions are not equivalent. Hence, either  $x \in (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$  and there exists a separated model  $\mathcal{M}' = (W', R', \star', V')$  such that  $\mathcal{M} \sim_p \mathcal{M}'$  and  $x \notin (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ , or  $x \notin (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$  and for all separated models  $\mathcal{M}' = (W', R', \star', V')$ , if  $\mathcal{M} \sim_p \mathcal{M}'$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ .

**Case ‘ $x \in (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$  and there exists a separated model  $\mathcal{M}' = (W', R', \star', V')$  such that  $\mathcal{M} \sim_p \mathcal{M}'$  and  $x \notin (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ .** Hence, there exists  $y \in W$  such that  $x(\alpha \parallel \beta)^{\mathcal{M}}y$  and  $y \in (\phi)^{\mathcal{M}}$ . Thus, there exists  $z, t, u, v \in W$  such that  $x \in z \star t$ ,  $y \in u \star v$ ,  $z(\alpha)^{\mathcal{M}}u$  and  $t(\beta)^{\mathcal{M}}v$ . Therefore,  $x(r_1)^{\mathcal{M}}z$ ,  $x(r_2)^{\mathcal{M}}t$ ,  $u(s_1)^{\mathcal{M}}y$  and  $v(s_2)^{\mathcal{M}}y$ . Since  $p$  does not occur in  $\alpha, \beta, \phi$ ,  $\mathcal{M} \sim_p \mathcal{M}'$ ,  $z(\alpha)^{\mathcal{M}}u$ ,  $t(\beta)^{\mathcal{M}}v$  and  $y \in (\phi)^{\mathcal{M}}$ ,  $z(\alpha)^{\mathcal{M}'}u$ ,  $t(\beta)^{\mathcal{M}'}v$  and  $y \in (\phi)^{\mathcal{M}'}$ .

**Subcase ‘ $y \in V'(p)$ ’.** Since  $y \in (\phi)^{\mathcal{M}'}$ ,  $y \in (\phi \wedge p)^{\mathcal{M}'}$ . Since  $x(r_1)^{\mathcal{M}}z$ ,  $z(\alpha)^{\mathcal{M}'}u$  and  $u(s_1)^{\mathcal{M}}y$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p))^{\mathcal{M}'}$ . Hence,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ : a contradiction.

**Subcase ‘ $y \notin V'(p)$ ’.** Since  $y \in (\phi)^{\mathcal{M}'}$ ,  $y \in (\phi \wedge \neg p)^{\mathcal{M}'}$ . Since  $x(r_2)^{\mathcal{M}}t$ ,  $t(\beta)^{\mathcal{M}'}v$  and  $v(s_2)^{\mathcal{M}}y$ ,  $x \in (\langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ . Hence,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ : a contradiction.

**Case ‘ $x \notin (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$  and for all separated models  $\mathcal{M}' = (W', R', \star', V')$ , if  $\mathcal{M} \sim_p \mathcal{M}'$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ .** Let  $\mathcal{M}'' = (W'', R'', \star'', V'')$  be a separated model such that  $\mathcal{M} \sim_p \mathcal{M}''$  and  $V''(p) = \{y: \text{there exists } z, t, u, v \in W \text{ such that } x \in z \star t, y \in u \star v \text{ and } t(\beta)^{\mathcal{M}}v\}$ . Since for all separated models  $\mathcal{M}' = (W', R', \star', V')$ , if  $\mathcal{M} \sim_p \mathcal{M}'$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}'}$ ,  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}''}$ . Hence, either  $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p))^{\mathcal{M}''}$ , or  $x \in (\langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}''}$ .

**Subcase ‘ $x \in (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p))^{\mathcal{M}''}$ ’.** Hence, there exists  $y, z, t, u, v \in W$  such that  $x \in z \star t$ ,  $y \in u \star v$ ,  $z(\alpha)^{\mathcal{M}''}u$ ,  $y \in (\phi)^{\mathcal{M}''}$  and  $y \in V''(p)$ . Since  $p$  does not occur in  $\alpha, \phi$  and  $\mathcal{M} \sim_p \mathcal{M}''$ ,  $z(\alpha)^{\mathcal{M}}u$  and  $y \in (\phi)^{\mathcal{M}}$ . Since  $\mathcal{M}$  is separated,  $x \in z \star t$ ,  $y \in u \star v$  and  $y \in V''(p)$ ,  $t(\beta)^{\mathcal{M}}v$ . Since  $x \in z \star t$ ,  $y \in u \star v$ , and  $z(\alpha)^{\mathcal{M}}u$ ,  $x(\alpha \parallel \beta)^{\mathcal{M}}y$ . Since  $y \in (\phi)^{\mathcal{M}}$ ,  $x \in (\langle \alpha \parallel \beta \rangle \phi)^{\mathcal{M}}$ : a contradiction.

**Subcase ‘ $x \in (\langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))^{\mathcal{M}''}$ ’.** Hence, there exists  $y, z, t, u, v \in W$  such that  $x \in z \star t$ ,  $y \in u \star v$ ,  $t(\beta)^{\mathcal{M}''}v$ ,  $y \in (\phi)^{\mathcal{M}''}$  and  $y \notin V''(p)$ . Since  $p$  does not occur in  $\beta$  and  $\mathcal{M} \sim_p \mathcal{M}''$ ,  $t(\beta)^{\mathcal{M}}v$ . Since  $x \in z \star t$  and  $y \in u \star v$ ,  $y \in V''(p)$ : a contradiction. ■

Let us temporarily add propositional quantifiers of the form  $\forall p$  to the language of  $PRSPDL_0$  for each atomic formula  $p$ . Such constructs allow to write formulas of the form  $\forall p \phi$ . In a model  $\mathcal{M} = (W, R, \star, V)$ , a formula of the form  $\forall p \phi$  is interpreted as the following subset of  $W$ :

- $(\forall p \phi)^{\mathcal{M}} = \bigcap \{(\phi)^{\mathcal{M}'} : \mathcal{M}' \text{ is a model such that } \mathcal{M} \sim_p \mathcal{M}'\}.$

A consequence of Proposition 13 is that the program construct  $(\cdot \parallel \cdot)$  of parallel composition becomes definable in a modal language strengthened by the introduction of propositional quantifiers. To see this, it suffices to consider the following

PROPOSITION 14

Let  $\alpha, \beta$  be programs,  $\phi$  be a formula and  $p$  an atomic formula. If  $p$  does not occur in  $\alpha, \beta, \phi$ , the formulas  $\langle \alpha \parallel \beta \rangle \phi$  and  $\forall p(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))$  are equally interpreted in all separated models.

PROOF. By Proposition 13. ■

In Sections 6 and 7, instead of using axioms to define the program operation of parallel composition in the language of  $PDL$  enlarged with propositional quantifiers, we add an unorthodox rule of proof that makes the canonical model standard for the program operation of parallel composition and we use large programs for the proof of the Truth Lemma. In our canonical model construction, large programs will constitute the main ingredients in the proofs of the Existence Lemma (Lemma 10) and the Truth Lemma (Lemma 11).



## 5 Definability

Now, about definability. We investigate the question whether our new constructs can be used to define the following elementary classes of frames: the class of all separated frames; the class of all rich frames; the class of all deterministic frames; the class of all serial frames.

PROPOSITION 15

The class of all separated frames is modally definable by the formulas  $\langle r_1 \rangle p \rightarrow [r_1]p$  and  $\langle r_2 \rangle p \rightarrow [r_2]p$ .

PROOF. Left to the reader. ■

PROPOSITION 16

The class of all rich frames is modally definable by the formula  $\langle r_1 \rangle \top \vee \langle r_2 \rangle \top$ .

PROOF. Left to the reader. ■

PROPOSITION 17

The class of all deterministic frames is modally definable by the formula  $\langle \top? \parallel \top? \rangle p \rightarrow p$ .

PROOF. Left to the reader. ■

PROPOSITION 18

The class of all serial frames is not modally definable.

PROOF. Suppose there exists a set  $\Sigma$  of formulas from the language of  $PRSPDL_0$  that modally defines the class of all serial frames. Let  $\mathcal{F} = (W, R, \star)$  and  $\mathcal{F}' = (W', R', \star')$  be the frames defined as follows:  $W = \{x_1, x_2\}$ ,  $R$  is the empty function,  $x_1 \star x_1 = \{x_1\}$ ,  $x_2 \star x_2 = \{x_2\}$  and otherwise  $\star$  is the empty function,  $W' = \{x'\}$ ,  $R'$  is the empty function and  $x' \star' x' = \{x'\}$ . Obviously,  $\mathcal{F}$  is not serial and  $\mathcal{F}'$  is serial. Since  $\Sigma$  modally defines the class of all serial frames,  $\mathcal{F} \not\models \Sigma$  and  $\mathcal{F}' \models \Sigma$ . Hence, there exists a formula  $\phi \in \Sigma$  such that  $\mathcal{F} \not\models \phi$ . Since  $\mathcal{F}' \models \Sigma$ ,  $\mathcal{F}' \models \phi$ . Since  $\mathcal{F} \not\models \phi$ , there exists a model  $\mathcal{M} = (W, R, \star, V)$  on  $\mathcal{F}$  such that either  $x_1 \notin (\phi)^{\mathcal{M}}$ , or  $x_2 \notin (\phi)^{\mathcal{M}}$ . Without loss of generality, assume  $x_1 \notin (\phi)^{\mathcal{M}}$ . Let  $\mathcal{M}' = (W', R', \star', V')$  be the model on  $\mathcal{F}'$  defined as follows:  $V'(p) = \{x' \mid x_1 \in V(p)\}$ , then  $\{x'\}$ , else  $\emptyset$  for every propositional variable  $p$ . Since  $\mathcal{F}' \models \phi$ ,  $x' \in (\phi)^{\mathcal{M}'}$ .

CLAIM: Let  $\alpha$  be a program and  $\psi$  be a formula from the language of  $PRSPDL_0$ . Then,

- $\text{not } x_1(\alpha)^{\mathcal{M}} x_2$ ;
- $x_1(\alpha)^{\mathcal{M}} x_1$  iff  $x'(\alpha)^{\mathcal{M}'} x'$ ;
- $x_1 \in (\psi)^{\mathcal{M}}$  iff  $x' \in (\psi)^{\mathcal{M}'}$ .

PROOF: By induction on  $\alpha$  and  $\psi$ . Left to the reader. ■

Since  $x_1 \notin (\phi)^{\mathcal{M}}$ ,  $x' \notin (\phi)^{\mathcal{M}'}$ : a contradiction.

## 6 Axiomatization

This section presents the axiomatization of  $PRSPDL_0$ . But before, we need to say more about the rule of proof (*FOR*). There is an important point we should make: (*FOR*) is an infinitary rule of proof, i.e. it has an infinite set of formulas as preconditions. In some ways, it is similar to the rules of proof for the program construct  $(\cdot \cap \cdot)$  of intersection from [3, 4]. Let us consider the following variant of (*FOR*):

(*FOR'*) from  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  where  $p$  is an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$ , infer  $\check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$ .

Obviously,  $(FOR')$  is a finitary rule of proof, i.e. it has a finite set of formulas—a singleton—as preconditions. How should we demonstrate the rules of proof  $(FOR)$  and  $(FOR')$  are equivalent in the sense that they are interchangeable? Let  $PRSPDL_0$  be the least set of formulas that contains the formulas (A1)–(A14) and that is closed under the rules of proof  $(MP)$ ,  $(N)$  and  $(FOR)$  and  $PRSPDL'_0$  be the least set of formulas that contains the formulas (A1)–(A14) and that is closed under the rules of proof  $(MP)$ ,  $(N)$  and  $(FOR')$ . We shall say that  $\phi$  is provable in  $PRSPDL_0$  iff  $\phi$  belongs to  $PRSPDL_0$ , whereas we shall say that  $\phi$  is provable in  $PRSPDL'_0$  iff  $\phi$  belongs to  $PRSPDL'_0$ . The infinitary nature of the rule of proof  $(FOR)$  implies that ‘ $PRSPDL_0$ -proofs’ can be of infinite length whereas the finitary nature of the rule of proof  $(FOR')$  implies that ‘ $PRSPDL'_0$ -proofs’ are always of finite length. More precisely, by definition of  $PRSPDL_0$  and  $PRSPDL'_0$ , for all formulas  $\phi$ ,

- $\phi$  belongs to  $PRSPDL_0$  iff there exists an ordinal  $\lambda$  and a  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  of formulas—called a  $\lambda$ -proof of  $\phi$  in  $PRSPDL_0$ —such that  $\psi_\lambda = \phi$  and for all  $\mu \leq \lambda$ , either  $\psi_\mu$  is one of the formulas (A1)–(A14), or  $\psi_\mu$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of one of the rules of proof  $(MP)$ ,  $(N)$  and  $(FOR)$ ;
- $\phi$  belongs to  $PRSPDL'_0$  iff there exists a non-negative integer  $\lambda$  and a  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  of formulas—called a  $\lambda$ -proof of  $\phi$  in  $PRSPDL'_0$ —such that  $\psi_\lambda = \phi$  and for all  $\mu \leq \lambda$ , either  $\psi_\mu$  is one of the formulas (A1)–(A14), or  $\psi_\mu$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of one of the rules of proof  $(MP)$ ,  $(N)$  and  $(FOR')$ .

Concerning  $PRSPDL'_0$ , we have the following.

LEMMA 1

Let  $\phi(p)$  be a formula and  $\lambda$  be a non-negative integer. If there exists a  $\lambda$ -proof of  $\phi(p)$  in  $PRSPDL'_0$ , for all atomic formulas  $q$  not occurring in  $\phi(p)$ , there exists a  $\lambda$ -proof of  $\phi(q)$  in  $PRSPDL'_0$ .

PROOF. By induction on  $\lambda$ . Left to the reader. ■

The rules of proof  $(FOR)$  and  $(FOR')$  are equivalent in the sense that they are interchangeable. More precisely,

PROPOSITION 19

Let  $\phi$  be a formula. The following conditions are equivalent:

- (1)  $\phi$  belongs to  $PRSPDL_0$ ;
- (2)  $\phi$  belongs to  $PRSPDL'_0$ .

PROOF. Suppose the above conditions are not equivalent. Hence, either  $\phi$  belongs to  $PRSPDL_0$  and  $\phi$  does not belong to  $PRSPDL'_0$ , or  $\phi$  does not belong to  $PRSPDL_0$  and  $\phi$  belongs to  $PRSPDL'_0$ .

**Case ‘ $\phi$  belongs to  $PRSPDL_0$  and  $\phi$  does not belong to  $PRSPDL'_0$ ’.** Hence, there exists an ordinal  $\lambda$  and a  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  of formulas such that  $\psi_\lambda = \phi$  and for all  $\mu \leq \lambda$ , either  $\psi_\mu$  is one of the formulas (A1)–(A14), or  $\psi_\mu$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of one of the rules of proof  $(MP)$ ,  $(N)$  and  $(FOR)$ . By induction on  $\lambda$ , let us verify that  $\phi$  belongs to  $PRSPDL'_0$ .

**Cases ‘ $\psi_\lambda$  is one of the formulas (A1)–(A14)’**, ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof  $(MP)$ ’ and ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof  $(N)$ ’. Left to the reader.

**Case ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof ( $FOR$ )’.** Hence, there exists an admissible form  $\check{\phi}$ , there exists programs  $\alpha, \beta$  and there exists a formula  $\psi$  such that for all atomic formulas  $p$  not occurring in  $\check{\phi}, \alpha, \beta, \psi$ ,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  is the formula  $\psi_\mu$  for some  $\mu < \lambda$  and  $\check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$  is the formula  $\psi_\lambda$ . Let  $p$  be an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$ . Thus,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  is the formula  $\psi_\mu$  for some  $\mu < \lambda$ . By induction hypothesis,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  belongs to  $PRSPDL'_0$ . Since  $p$  does not occur in  $\check{\phi}, \alpha, \beta, \psi$ ,  $\psi_\lambda$  belongs to  $PRSPDL'_0$ : a contradiction.

**Case ‘ $\phi$  does not belong to  $PRSPDL_0$  and  $\phi$  belongs to  $PRSPDL'_0$ ’.** Hence, there exists a non-negative integer  $\lambda$  and a  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  of formulas such that  $\psi_\lambda = \phi$  and for all  $\mu \leq \lambda$ , either  $\psi_\mu$  is one of the formulas (A1)–(A14), or  $\psi_\mu$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of one of the rules of proof ( $MP$ ), ( $N$ ) and ( $FOR$ ). By induction on  $\lambda$ , let us verify that  $\phi$  belongs to  $PRSPDL_0$ .

**Cases ‘ $\psi_\lambda$  is one of the formulas (A1)–(A14)’**, ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof ( $MP$ )’ and ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof ( $N$ )’.

Left to the reader.

**Case ‘ $\psi_\lambda$  is obtained from previous formulas in the  $\lambda$ -termed sequence  $(\psi_\mu)_{\mu \leq \lambda}$  by means of the rule of proof ( $FOR$ )’.** Hence, there exists an admissible form  $\check{\phi}$ , there exists programs  $\alpha, \beta$  and there exists a formula  $\psi$  such that  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$ —where  $p$  is an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$ —is the formula  $\psi_\mu$  for some  $\mu < \lambda$  and  $\check{\phi}(\langle \alpha \parallel \beta \rangle \psi)$  is the formula  $\psi_\lambda$ . Thus, there exists a  $\mu$ -proof of  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg p))$  in  $PRSPDL'_0$ . By Lemma 1, since  $p$  is an atomic formula not occurring in  $\check{\phi}, \alpha, \beta, \psi$ , for all atomic formulas  $q$  not occurring in  $\check{\phi}, \alpha, \beta, \psi$ , there exists a  $\mu$ -proof of  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge q) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg q))$  in  $PRSPDL'_0$ . By induction hypothesis, for all atomic formulas  $q$  not occurring in  $\check{\phi}, \alpha, \beta, \psi$ ,  $\check{\phi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\psi \wedge q) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\psi \wedge \neg q))$  belongs to  $PRSPDL_0$ . Therefore,  $\psi_\lambda$  belongs to  $PRSPDL_0$ : a contradiction. ■

Hence, as long as one is interested in the notion of derivability, ( $FOR$ ) and ( $FOR'$ ) are equivalent. To see how the rule of proof ( $FOR$ ) works, let us demonstrate the following

## LEMMA 2

Let  $\alpha(\phi?)$  be a program. For all formulas  $\psi$ , if  $\phi \rightarrow \psi \in PRSPDL_0$ , for all formulas  $\chi$ ,  $\langle \alpha(\phi?) \rangle \chi \rightarrow \langle \alpha(\phi?) \rangle \chi \in PRSPDL_0$ .

**PROOF.** By induction on  $\alpha(\phi?)$ .

**Cases ‘ $\alpha(\phi?) = a$ ’, ‘ $\alpha(\phi?) = \theta?$ ’, ‘ $\alpha(\phi?) = s_1$ ’, ‘ $\alpha(\phi?) = s_2$ ’, ‘ $\alpha(\phi?) = r_1$ ’, ‘ $\alpha(\phi?) = r_2$ ’, ‘ $\alpha(\phi?) = \beta(\phi?); \gamma$ ’ and ‘ $\alpha(\phi?) = \beta; \gamma(\phi?)$ ’.** Left to the reader.

**Case ‘ $\alpha(\phi?) = \beta(\phi?) \parallel \gamma$ ’.** By (A10),  $\langle \beta(\phi?) \parallel \gamma \rangle \chi \rightarrow \langle r_1 \rangle \langle \beta(\phi?) \rangle \langle s_1 \rangle (\chi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\chi \wedge \neg p) \in PRSPDL_0$  for every atomic formula  $p$  not occurring in  $\beta(\phi?), \gamma, \psi, \chi$ . By induction hypothesis,  $\langle \beta(\phi?) \parallel \gamma \rangle \chi \rightarrow \langle r_1 \rangle \langle \beta(\psi?) \rangle \langle s_1 \rangle (\chi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\chi \wedge \neg p) \in PRSPDL_0$  for every atomic formula  $p$  not occurring in  $\beta(\phi?), \gamma, \psi, \chi$ . Hence,  $[\langle \beta(\phi?) \parallel \gamma \rangle \chi?; \neg(\langle r_1 \rangle \langle \beta(\psi?) \rangle \langle s_1 \rangle (\chi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\chi \wedge \neg p))] \perp \in PRSPDL_0$  and  $[(\langle \beta(\phi?) \parallel \gamma \rangle \chi?; \neg\sharp?)(\langle r_1 \rangle \langle \beta(\psi?) \rangle \langle s_1 \rangle (\chi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\chi \wedge \neg p))] \perp \in PRSPDL_0$  for every atomic formula  $p$  not occurring in  $\beta(\phi?), \gamma, \psi, \chi$ . By ( $FOR$ ),  $[(\langle \beta(\phi?) \parallel \gamma \rangle \chi?; \neg\sharp?)(\langle \beta(\psi?) \parallel$

$\gamma \rangle \chi \rangle \perp \in PRSPDL_0$ ,  $[\langle \beta(\phi?) \parallel \gamma \rangle \chi?; \neg \langle \beta(\psi?) \parallel \gamma \rangle \chi?] \perp \in PRSPDL_0$  and  $[\langle \beta(\phi?) \parallel \gamma \rangle \chi?] \langle \beta(\psi?) \parallel \gamma \rangle \chi \in PRSPDL_0$ . Thus,  $\langle \beta(\phi?) \parallel \gamma \rangle \chi \rightarrow \langle \beta(\psi?) \parallel \gamma \rangle \chi \in PRSPDL_0$ .

**Case ‘ $\alpha(\phi?) = \beta \parallel \gamma(\phi?)$ ’.** Similar to the case ‘ $\alpha(\phi?) = \beta(\phi?) \parallel \gamma$ ’.

Having said this, now, let us establish the soundness for  $PRSPDL_0$ :

PROPOSITION 20 (Soundness for  $PRSPDL_0$ )

Let  $\phi$  be a formula. If  $\phi \in PRSPDL_0$ ,  $\phi$  is valid in the class of all separated frames.

PROOF. By Propositions 4–7.

The completeness for  $PRSPDL_0$  is more difficult to establish and we defer proving it till next section. In the meantime, it is well worth noting that for all separated models  $\mathcal{M} = (W, R, \star, V)$  and for all  $x \in W$ ,  $\{\phi : x \in (\phi)^{\mathcal{M}}\}$  is a set of formulas that contains  $PRSPDL_0$  and that is closed under the rule of proof (MP). Now, we introduce theories. A set  $S$  of formulas is said to be a theory iff  $PRSPDL_0 \subseteq S$  and  $S$  is closed under the rules of proof (MP) and (FOR). We will use  $S, T, \dots$  for theories. Obviously, the least theory is  $PRSPDL_0$  and the greatest theory is the set of all formulas. We will use the following property of theories without explicit reference.

LEMMA 3

Let  $S$  be a theory. The following conditions are equivalent:

- $S$  is equal to the set of all formulas;
- there exists a formula  $\phi$  such that  $\phi \in S$  and  $\neg\phi \in S$ ;
- $\perp \in S$ .

PROOF. Left to the reader.

We shall say that a theory  $S$  is consistent iff for all formulas  $\phi$ , either  $\phi \notin S$ , or  $\neg\phi \notin S$ . By Lemma 3, there is only one inconsistent theory: the set of all formulas. A theory  $S$  is said to be maximal iff for all formulas  $\phi$ , either  $\phi \in S$ , or  $\neg\phi \in S$ . In Section 7, the canonical frame of  $PRSPDL_0$  and the canonical model of  $PRSPDL_0$  will be based on the set of all maximal consistent theories, whereas in the classical literature [7, Chapter 4], canonical frames and canonical models are based on the set of all maximal consistent sets of formulas. The truth is that every maximal consistent theory is a maximal consistent set of formulas in the classical sense, whereas every maximal consistent set of formulas closed under the rule of proof (FOR) is a maximal consistent theory. Hence,

LEMMA 4

Let  $S$  be a maximal consistent theory. We have:

- $\perp \notin S$ ;
- for all formulas  $\phi$ ,  $\neg\phi \in S$  iff  $\phi \notin S$ ;
- for all formulas  $\phi, \psi$ ,  $\phi \vee \psi \in S$  iff either  $\phi \in S$ , or  $\psi \in S$ .

PROOF. Left to the reader.

If  $\alpha$  is a program,  $\phi$  is a formula and  $S$  is a theory, let  $[\alpha]S = \{\phi : [\alpha]\phi \in S\}$  and  $S + \phi = \{\psi : \phi \rightarrow \psi \in S\}$ . Sets of the form  $[a]S$  will be used while defining the canonical relations  $R_c(a)$  in the canonical frame of  $PRSPDL_0$ . Sets of the form  $S + \phi$  will be used while demonstrating Lemma 7. We have the following.

#### LEMMA 5

Let  $S$  be a theory. For all programs  $\alpha$  and for all formulas  $\phi$ , we have:

- (1)  $[\phi?]S = S + \phi$ ;
- (2)  $[\alpha]S$  is a theory;
- (3)  $S + \phi$  is a theory;
- (4)  $S + \phi$  is the least theory containing  $S$  and  $\phi$ ;
- (5)  $S + \phi$  is consistent iff  $\neg\phi \notin S$ .

PROOF. (1). By (A3).

(2). By the rule of proof (N),  $[\alpha]S$  contains  $PRSPDL_0$ . By (A2),  $[\alpha]S$  is closed under the rule of proof (MP). We demonstrate  $[\alpha]S$  is closed under the rule of proof (FOR). Suppose  $\check{\phi}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\psi \wedge \neg p)) \in [\alpha]S$  for all atomic formulas  $p$  not occurring in  $\phi, \beta, \gamma, \psi$ . Hence,  $[\alpha]\check{\phi}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\psi \wedge \neg p)) \in S$ ,  $[\alpha; \neg\check{\phi}^?(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\psi \wedge \neg p))] \perp \in S$  and  $[(\alpha; \neg\check{\phi}^?)(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\psi \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\psi \wedge \neg p))] \perp \in S$  for all atomic formulas  $p$  not occurring in  $\alpha, \phi, \beta, \gamma, \psi$ . Since  $S$  is closed under the rule of proof (FOR),  $[(\alpha; \neg\check{\phi}^?)(\langle \beta \parallel \gamma \rangle \psi)] \perp \in S$ ,  $[\alpha; \neg\check{\phi}^?(\langle \beta \parallel \gamma \rangle \psi)] \perp \in S$  and  $[\alpha]\check{\phi}^?(\langle \beta \parallel \gamma \rangle \psi) \in S$ . Thus,  $\phi(\langle \beta \parallel \gamma \rangle \psi) \in [\alpha]S$ .

(3). By (1) and (2).

(4). Left to the reader.

(5). By Lemma 4. ■

In the classical literature, three Lemmas support the canonical model construction: the Lindenbaum Lemma [7, Lemma 4.17], the Existence Lemma [7, Lemma 4.20] and the Truth Lemma [7, Lemma 4.21]. Our canonical model construction is also built on the same three Lemmas. Nevertheless, the fact that the canonical frame of  $PRSPDL_0$  and the canonical model of  $PRSPDL_0$  are based on the set of all maximal consistent theories creates some subtleties that we will now attack from the front. The Lindenbaum Lemma will say that every consistent theory can be extended to a maximal consistent theory. Hence, in a first setting, we have to learn how to extend a consistent theory by means of a formula.

#### LEMMA 6

Let  $S$  be a theory. If  $S$  is consistent, for all formulas  $\phi$ , either  $S + \phi$  is consistent, or there exists a formula  $\psi$  such that the following conditions are satisfied:

- $S + \psi$  is consistent;
- $\psi \rightarrow \neg\phi \in PRSPDL_0$ ;
- if  $\phi$  is in the form  $\check{\chi}(\langle \alpha \parallel \beta \rangle \theta)$  of a conclusion of the rule of proof (FOR), there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\psi \rightarrow \neg\check{\chi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\theta \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\theta \wedge \neg p)) \in PRSPDL_0$ .

PROOF. Suppose  $S$  is consistent and  $\phi$  is a formula such that  $S + \phi$  is not consistent. By Lemma 5,  $\neg\phi \in S$ . Obviously, there are finitely many, say  $k \geq 0$ , representations of  $\phi$  in the form of a conclusion of the rule of proof (FOR):  $\check{\chi}_1(\langle \alpha_1 \parallel \beta_1 \rangle \theta_1), \dots, \check{\chi}_k(\langle \alpha_k \parallel \beta_k \rangle \theta_k)$ . We define by induction a sequence  $(\psi_0, \dots, \psi_k)$  of formulas such that for all  $l \in \mathbb{N}$ , if  $l \leq k$ , the following conditions are satisfied:  $S + \psi_l$  is consistent;  $\psi_l \rightarrow \neg\phi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l$ , there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\psi_l \rightarrow \neg\check{\chi}_m(\langle r_1 \rangle \langle \alpha_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \beta_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . First, let  $\psi_0 = \neg\phi$ . Obviously, the following conditions are satisfied:  $S + \psi_0$  is consistent;  $\psi_0 \rightarrow \neg\phi \in PRSPDL_0$ . Secondly, let  $l \geq 1$  be such that  $l \leq k$  and the formulas  $\psi_0, \dots, \psi_{l-1}$  have already

been defined. Hence,  $S + \psi_{l-1}$  is consistent;  $\psi_{l-1} \rightarrow \neg\phi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l-1$ , there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\psi_{l-1} \rightarrow \neg\check{\chi}_m(\langle r_1 \rangle \langle \alpha_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \beta_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . Thirdly, since  $S + \psi_{l-1}$  is consistent and  $\psi_{l-1} \rightarrow \neg\phi \in PRSPDL_0$ ,  $\phi \notin S + \psi_{l-1}$ . Since  $S + \psi_{l-1}$  is closed under the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\check{\chi}_l(\langle r_1 \rangle \langle \alpha_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \beta_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p)) \notin S + \psi_{l-1}$ . Let  $\psi_l = \psi_{l-1} \wedge \neg\check{\chi}_l(\langle r_1 \rangle \langle \alpha_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \beta_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p))$ . Obviously, the following conditions are satisfied:  $S + \psi_l$  is consistent;  $\psi_l \rightarrow \neg\phi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l$ , there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\psi_l \rightarrow \neg\check{\chi}_m(\langle r_1 \rangle \langle \alpha_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \beta_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . Finally, the reader may easily verify that the following conditions are satisfied:  $S + \psi_k$  is consistent;  $\psi_k \rightarrow \neg\phi \in PRSPDL_0$ ; if  $\phi$  is in the form  $\check{\chi}(\langle \alpha \parallel \beta \rangle \theta)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\phi$  such that  $\psi_k \rightarrow \neg\check{\chi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\theta \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\theta \wedge \neg p)) \in PRSPDL_0$ . ■

Now, knowing how to extend a consistent theory by means of a formula, we can demonstrate the Lindenbaum Lemma.

LEMMA 7 (Lindenbaum Lemma)

Let  $S$  be a theory. If  $S$  is consistent, there exists a maximal consistent theory containing  $S$ .

PROOF. Suppose  $S$  is consistent. Since there are countably many formulas, there exists an enumeration  $\phi_1, \phi_2, \dots$  of the set of all formulas. Let  $T_0, T_1, \dots$  be the sequence of consistent theories inductively defined as follows. First, let  $T_0 = S$ . Obviously,  $T_0$  is consistent. Secondly, let  $n \geq 1$  be such that consistent theories  $T_0, \dots, T_{n-1}$  have already been defined. Thirdly, by Lemma 6, either  $T_{n-1} + \phi_n$  is consistent, or there exists a formula  $\psi$  such that the following conditions are satisfied:  $T_{n-1} + \psi$  is consistent;  $\psi \rightarrow \neg\phi_n \in PRSPDL_0$ ; if  $\phi_n$  is in the form  $\check{\chi}(\langle \alpha \parallel \beta \rangle \theta)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\check{\chi}, \alpha, \beta, \theta$  such that  $\psi \rightarrow \neg\check{\chi}(\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\theta \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\theta \wedge \neg p)) \in PRSPDL_0$ . In the former case, let  $T_n = T_{n-1} + \phi_n$ . In the latter case, let  $T_n = T_{n-1} + \psi$ . Obviously,  $T_n$  is consistent. Finally, the reader may easily verify that  $T_0 \cup T_1 \cup \dots$  is a maximal consistent theory containing  $S$ . ■

## 7 Completeness

This section proves the completeness of  $PRSPDL_0$ . The canonical frame of  $PRSPDL_0$  is the 3-tuple  $\mathcal{F}_c = (W_c, R_c, \star_c)$  where  $W_c$  is the set of all maximal consistent theories,  $R_c$  is the function from the set of all atomic programs into the set of all binary relations between maximal consistent theories defined by  $SR_c(a)T$  iff  $[a]S \subseteq T$  and  $\star_c$  is the function from the set of all pairs of maximal consistent theories into the set of all sets of maximal consistent theories defined by  $U \in S \star_c T$  iff  $[s_1]S \subseteq U$  and  $[s_2]T \subseteq U$ . We firstly demonstrate the following

LEMMA 8

$\mathcal{F}_c$  is separated.

PROOF. Suppose  $\mathcal{F}_c$  is not separated. Hence, there exists a maximal consistent theory  $S$  such that  $\text{card}(\star_c(S)) \geq 2$ . Thus, there exists maximal consistent theories  $T, U, V, W$  such that  $S \in T \star_c U$ ,  $S \in V \star_c W$  and either  $T \neq V$ , or  $U \neq W$ . Without loss of generality, suppose  $T \neq V$ . Hence, there exists a formula  $\phi$  such that  $\phi \in T$  and  $\phi \notin V$ . Since  $S \in T \star_c U$  and  $S \in V \star_c W$ ,  $[s_1]T \subseteq S$  and  $[s_1]V \subseteq S$ . By (A4),  $[r_1]S \subseteq T$  and  $[r_1]S \subseteq V$ . Since  $\phi \in T$ ,  $\langle r_1 \rangle \phi \in S$ . By (A13),  $[r_1]\phi \in S$ . Since  $[r_1]S \subseteq V$ ,  $\phi \in V$ : a contradiction. ■



The canonical model of  $PRSPDL_0$  is the 4-tuple  $\mathcal{M}_c = (W_c, R_c, \star_c, V_c)$  where  $V_c: p \mapsto V_c(p) \subseteq W_c$  is the canonical valuation of  $PRSPDL_0$ , i.e. the function from the set of all atomic formulas into the set of all sets of maximal consistent theories defined by  $S \in V_c(p)$  iff  $p \in S$ . In our canonical model construction, the ordinary form of the Existence Lemma would be as follows: for all programs  $\alpha$ , for all formulas  $\phi$  and for all maximal consistent theories  $S$ , if  $[\alpha]\phi \notin S$ , there exists a maximal consistent theory  $T$  such that  $[\alpha]S \subseteq T$  and  $\phi \notin T$ . Nevertheless, it happens that the proof of our Truth Lemma (Lemma 11) needs a stronger form of the Existence Lemma. This stronger form requires the use of a new modal concept: large programs. For all consistent theories  $S$ , let  $\bar{S}$  be a new symbol at our disposal. Now, the set of all large programs is inductively defined as follows:

- $A \longrightarrow a \mid \bar{S} \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (A; B) \mid (A \parallel B)$ .

We will use  $A, B, \dots$  for large programs. Let us be clear that each large program is a finite string of symbols coming from an uncountable alphabet. It follows that there are uncountably many large programs. For convenience, we omit the parentheses in accordance with the standard rules. It is essential that large programs are built up from atomic programs, symbols for consistent theories, storing constructs and recovering constructs by means of the constructs  $(\cdot; \cdot)$  and  $(\cdot \parallel \cdot)$ . Let  $A(\bar{S}_1, \dots, \bar{S}_n)$  be a large program with  $(\bar{S}_1, \dots, \bar{S}_n)$  a sequence of some of its symbols for consistent theories. The result of the replacement of  $\bar{S}_1, \dots, \bar{S}_n$  in their places with  $\bar{T}_1, \dots, \bar{T}_n$  is another large program which will be denoted  $A(\bar{T}_1, \dots, \bar{T}_n)$ . A large program  $A(\bar{S}_1, \dots, \bar{S}_n)$  with  $(\bar{S}_1, \dots, \bar{S}_n)$  the sequence of all its symbols for consistent theories will be defined to be maximal if the theories  $S_1, \dots, S_n$  are maximal. In the canonical model, every large program will be interpreted as a binary relation over the set of all maximal consistent theories. To define such a binary relation, one needs to view each large program as a set of programs. In this respect, the kernel function  $\ker: A \mapsto \ker(A) \subseteq PR$  is inductively defined as follows:

- $\ker(a) = \{a\}$ ;
- $\ker(\bar{S}) = \{\phi?: \phi \in S\}$ ;
- $\ker(s_1) = \{s_1\}$ ;
- $\ker(s_2) = \{s_2\}$ ;
- $\ker(r_1) = \{r_1\}$ ;
- $\ker(r_2) = \{r_2\}$ ;
- $\ker(A; B) = \{\alpha; \beta: \alpha \in \ker(A) \text{ and } \beta \in \ker(B)\}$ ;
- $\ker(A \parallel B) = \{\alpha \parallel \beta: \alpha \in \ker(A) \text{ and } \beta \in \ker(B)\}$ .

Lemma 9 will be put to good use in the proof of the Existence Lemma.

#### LEMMA 9

Let  $\alpha$  be a program. For all maximal consistent theories  $S$  and for all formulas  $\phi$ , if  $\langle \alpha(\phi?) \rangle \top \in S$ , for all formulas  $\psi$ , we have: either  $\langle \alpha((\phi \wedge \psi)?) \rangle \top \in S$ , or there exists a formula  $\chi$  such that the following conditions are satisfied:

- $\langle \alpha((\phi \wedge \chi)?) \rangle \top \in S$ ;
- $\chi \rightarrow \neg\psi \in PRSPDL_0$ ;
- if  $\psi$  is in the form  $\check{\tau}(\langle \beta \parallel \gamma \rangle \theta)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi$  such that  $\chi \rightarrow \neg\check{\tau}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\theta \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\theta \wedge \neg p)) \in PRSPDL_0$ .

PROOF. Suppose  $S$  is a maximal consistent theory and  $\phi$  is a formula such that  $\langle \alpha(\phi?) \rangle \top \in S$  and  $\psi$  is a formula such that  $\langle \alpha((\phi \wedge \psi)?) \rangle \top \notin S$ . By (A11),  $\langle \alpha((\phi \wedge \neg\psi)?) \rangle \top \in S$ . Obviously,

there are finitely many, say  $k \geq 0$ , representations of  $\psi$  in the form of a conclusion of the rule of proof (*FOR*):  $\check{\tau}_1(\langle \beta_1 \parallel \gamma_1 \rangle \theta_1), \dots, \check{\tau}_k(\langle \beta_k \parallel \gamma_k \rangle \theta_k)$ . We define by induction a sequence  $(\chi_0, \dots, \chi_k)$  of formulas such that for all  $l \in \mathbb{N}$ , if  $l \leq k$ , the following conditions are satisfied:  $\langle \alpha((\phi \wedge \chi_l)?) \rangle \top \in S$ ;  $\chi_l \rightarrow \neg\psi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l$ , there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi_l$  such that  $\chi_l \rightarrow \neg\check{\tau}_m(\langle r_1 \rangle \langle \beta_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \gamma_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . First, let  $\chi_0 = \neg\psi$ . Obviously, the following conditions are satisfied:  $\langle \alpha((\phi \wedge \chi_0)?) \rangle \top \in S$ ;  $\chi_0 \rightarrow \neg\psi \in PRSPDL_0$ . Secondly, let  $l \geq 1$  be such that  $l \leq k$  and the formulas  $\chi_0, \dots, \chi_{l-1}$  have already been defined. Hence,  $\langle \alpha((\phi \wedge \chi_{l-1})?) \rangle \top \in S$ ;  $\chi_{l-1} \rightarrow \neg\psi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l-1$ , there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi_{l-1}$  such that  $\chi_{l-1} \rightarrow \neg\check{\tau}_m(\langle r_1 \rangle \langle \beta_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \gamma_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . Third, by Lemma 2, since  $\langle \alpha((\phi \wedge \chi_{l-1})?) \rangle \top \in S$  and  $\chi_{l-1} \rightarrow \neg\psi \in PRSPDL_0$ ,  $\langle \alpha((\phi \wedge \chi_{l-1} \wedge \neg\psi)?) \rangle \top \in S$ . Thus,  $\langle \alpha((\phi \wedge \chi_{l-1} \wedge \neg\psi)?) \rangle \perp \notin S$ ,  $\langle \alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg\psi?] \perp?) \rangle \perp \notin S$  and  $\langle \alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg\psi?] \perp?) \rangle \perp (\psi) \notin S$ . Since  $S$  is closed under the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi_{l-1}$  such that  $\langle \alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg\psi?] \perp?) \rangle \perp (\check{\tau}_l(\langle r_1 \rangle \langle \beta_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \gamma_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p))) \notin S$ . Therefore,  $\langle \alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg(\check{\tau}_l(\langle r_1 \rangle \langle \beta_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \gamma_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p)))?] \perp?) \rangle \perp \notin S$ ,  $\langle \alpha((\phi \wedge \chi_{l-1} \wedge \neg(\check{\tau}_l(\langle r_1 \rangle \langle \beta_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \gamma_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p))))?) \rangle \perp \notin S$  and  $\langle \alpha((\phi \wedge \chi_{l-1} \wedge \neg\check{\tau}_l(\langle r_1 \rangle \langle \beta_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \gamma_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p)))?) \rangle \top \in S$ . Let  $\chi_l = \chi_{l-1} \wedge \neg\check{\tau}_l(\langle r_1 \rangle \langle \beta_l \rangle \langle s_1 \rangle (\theta_l \wedge p) \vee \langle r_2 \rangle \langle \gamma_l \rangle \langle s_2 \rangle (\theta_l \wedge \neg p))$ . Obviously, the following conditions are satisfied:  $\langle \alpha((\phi \wedge \chi_l)?) \rangle \top \in S$ ;  $\chi_l \rightarrow \neg\psi \in PRSPDL_0$ ; for all  $m \in \mathbb{N}$ , if  $1 \leq m \leq l$ , there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi_l$  such that  $\chi_l \rightarrow \neg\check{\tau}_m(\langle r_1 \rangle \langle \beta_m \rangle \langle s_1 \rangle (\theta_m \wedge p) \vee \langle r_2 \rangle \langle \gamma_m \rangle \langle s_2 \rangle (\theta_m \wedge \neg p)) \in PRSPDL_0$ . Finally, the reader may easily verify that the following conditions are satisfied:  $\langle \alpha((\phi \wedge \chi_k)?) \rangle \top \in S$ ;  $\chi_k \rightarrow \neg\psi \in PRSPDL_0$ ; if  $\psi$  is in the form  $\check{\tau}(\langle \beta \parallel \gamma \rangle \theta)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\alpha, \phi, \psi, \chi_k$  such that  $\chi_k \rightarrow \neg\check{\tau}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\theta \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\theta \wedge \neg p)) \in PRSPDL_0$ .  $\blacksquare$

Now, we can demonstrate the Existence Lemma and the Truth Lemma.

#### LEMMA 10 (Existence Lemma)

Let  $\alpha$  be a program and  $\phi$  be a formula. For all maximal consistent theories  $S$ , if  $\langle \alpha \rangle \phi \notin S$ , there exists a maximal program  $A$  and there exists a maximal consistent theory  $T$  such that  $f(\alpha) \in \ker(A)$ , for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $\langle \beta \rangle S \subseteq T$  and  $\phi \notin T$ .

PROOF. Suppose there exists a maximal consistent theory  $S$  such that  $\langle \alpha \rangle \phi \notin S$ . Since  $S$  is maximal,  $\langle \alpha \rangle \neg\phi \in S$ . By (A12),  $\langle f(\alpha) \rangle \neg\phi \in S$ . Without loss of generality, suppose  $f(\alpha)$  contains exactly one test, say  $\psi?$ . Since  $\langle f(\alpha) \rangle \neg\phi \in S$ ,  $\langle f(\alpha) \rangle (\psi?); \neg\phi? \rangle \top \in S$ . Since there are countably many formulas, there exists an enumeration  $\chi_1, \chi_2, \dots$  of the set of all formulas. Let  $\theta^0, \theta^1, \dots$  and  $\tau^0, \tau^1, \dots$  be the sequences of formulas inductively defined as follows such that for all  $n \in \mathbb{N}$ ,  $\langle f(\alpha) \rangle (\theta^n?); \tau^n? \rangle \top \in S$ . First, let  $\theta^0 = \psi$  and  $\tau^0 = \neg\phi$ . Obviously,  $\langle f(\alpha) \rangle (\theta^0?); \tau^0? \rangle \top \in S$ . Secondly, let  $n \geq 1$  be such that formulas  $\theta^0, \dots, \theta^{n-1}$  and  $\tau^0, \dots, \tau^{n-1}$  have already been defined. Hence,  $\langle f(\alpha) \rangle (\theta^{n-1}?); \tau^{n-1}? \rangle \top \in S$ . Thirdly, by Lemma 9, either  $\langle f(\alpha) \rangle ((\theta^{n-1} \wedge \chi_n)?); \tau^{n-1}? \rangle \top \in S$ , or there exists a formula  $\mu$  such that the following conditions are satisfied:  $\langle f(\alpha) \rangle ((\theta^{n-1} \wedge \mu)?); \tau^{n-1}? \rangle \top \in S$ ;  $\mu \rightarrow \neg\chi_n \in PRSPDL_0$ ; if  $\chi_n$  is in the form  $\check{\omega}(\langle \beta \parallel \gamma \rangle \nu)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\alpha, \theta^{n-1}, \tau^{n-1}, \chi_n, \mu$  such that  $\mu \rightarrow \neg\check{\omega}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\nu \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\nu \wedge \neg p)) \in PRSPDL_0$ . In the former case, let  $\theta^n = \theta^{n-1} \wedge \chi_n$ . In the latter case, let  $\theta^n = \theta^{n-1} \wedge \mu$ . Obviously,  $\langle f(\alpha) \rangle (\theta^n?); \tau^{n-1}? \rangle \top \in S$ . By Lemma 9, either  $\langle f(\alpha) \rangle (\theta^n?); (\tau^{n-1} \wedge \chi_n)? \rangle \top \in S$ , or there exists a formula  $\mu'$  such that the following conditions are satisfied:  $\langle f(\alpha) \rangle (\theta^n?); (\tau^{n-1} \wedge \mu')? \rangle \top \in S$ ;  $\mu' \rightarrow \neg\chi_n \in PRSPDL_0$ ; if  $\chi_n$  is in the form  $\check{\omega}(\langle \beta \parallel \gamma \rangle \nu)$  of a conclusion of the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\alpha, \theta^n, \tau^{n-1}, \chi_n, \mu$  such that  $\mu' \rightarrow \neg\check{\omega}(\langle r_1 \rangle \langle \beta \rangle \langle s_1 \rangle (\nu \wedge p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (\nu \wedge \neg p)) \in PRSPDL_0$ .  $\blacksquare$

$p) \vee \langle r_2 \rangle \langle \gamma \rangle \langle s_2 \rangle (v \wedge \neg p)) \in PRSPDL_0$ . In the former case, let  $\tau^n = \tau^{n-1} \wedge \chi_n$ . In the latter case, let  $\tau^n = \tau^{n-1} \wedge \mu'$ . Obviously,  $\langle f(\alpha)(\theta^n?) ; \tau^n? \rangle \top \in S$ . Finally, the reader may easily verify that  $T = \bigcup \{PRSPDL_0 + \theta^n : n \in \mathbb{N}\}$  and  $U = \bigcup \{PRSPDL_0 + \tau^n : n \in \mathbb{N}\}$  are maximal consistent theories such that  $f(\alpha) \in \ker(f(\alpha)(\bar{T}))$ , for all programs  $\beta$ , if  $\beta \in \ker(f(\alpha)(\bar{T}))$ ,  $[\beta]S \subseteq U$  and  $\phi \notin U$ . ■

LEMMA 11 (Truth Lemma)

Let  $\alpha$  be a program and  $\phi$  be a formula.

- For all maximal consistent theories  $S, T$ ,  $S(\alpha)^{\mathcal{M}_c} T$  iff there exists a maximal program  $A$  such that  $f(\alpha) \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ .
- for all maximal consistent theories  $S$ ,  $S \in (\phi)^{\mathcal{M}_c}$  iff  $\phi \in S$ .

PROOF. Let  $P(\cdot)$  be the property about programs and formulas defined as follows:

- for all programs  $\alpha$ ,  $P(\alpha)$  iff for all maximal consistent theories  $S, T$ ,  $S(\alpha)^{\mathcal{M}_c} T$  iff there exists a maximal program  $A$  such that  $f(\alpha) \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ ;
- for all formulas  $\phi$ ,  $P(\phi)$  iff for all maximal consistent theories  $S$ ,  $S \in (\phi)^{\mathcal{M}_c}$  iff  $\phi \in S$ .

The proof that  $P(\cdot)$  holds for all programs and for all formulas will be done by induction on the formation of programs and formulas.

**Hypothesis.** Let  $\alpha$  be a program and  $\phi$  be a formula such that for all expressions  $exp$  (either a program, or a formula), if  $exp$  is an expression strictly occurring either in  $\alpha$ , or in  $\phi$ ,  $P(exp)$  holds.

**Step.** We demonstrate  $P(\alpha)$  and  $P(\phi)$  hold.

**Case ‘ $\alpha = a$ ’.** Left to the reader.

**Case ‘ $\alpha = \psi?$ ’.** Let  $S, T$  be maximal consistent theories.

- Suppose  $S(\psi?)^{\mathcal{M}_c} T$ . We demonstrate there exists a maximal program  $A$  such that  $\psi? \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . Since  $S(\psi?)^{\mathcal{M}_c} T$ ,  $S = T$  and  $T \in (\psi)^{\mathcal{M}_c}$ . Since  $P(\psi)$ ,  $\psi \in T$ . Since  $S = T$ ,  $\psi \in S$ . Hence,  $\psi? \in \ker(\bar{S})$ . Now, let  $\chi? \in \ker(\bar{S})$ . Thus,  $\chi \in S$ . Therefore,  $[\chi?]S = S$ . Since  $S = T$ ,  $[\chi?]S \subseteq T$ . Consequently, for all programs  $\beta$ , if  $\beta \in \ker(\bar{S})$ ,  $[\beta]S \subseteq T$ . Since  $\psi? \in \ker(\bar{S})$ , it suffices to take  $A = \bar{S}$ .
- Suppose there exists a maximal program  $A$  such that  $\psi? \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . We demonstrate  $S(\psi?)^{\mathcal{M}_c} T$ . Since  $\psi? \in \ker(A)$ , there exists a maximal consistent theory  $U$  such that  $\psi \in U$  and  $A = \bar{U}$ . Since for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ , for all formulas  $\chi$ , if  $\chi \in U$ ,  $[\chi?]S \subseteq T$ . Since  $\psi \in U$  and  $\top \in U$ ,  $[\psi?]S \subseteq T$  and  $[\top?]S \subseteq T$ . Since  $\top \in S$ ,  $[\top?]S = S$ . Since  $[\top?]S \subseteq T$ ,  $S \subseteq T$ . Since  $S$  is maximal and  $T$  is consistent,  $S = T$ . Since  $[\psi?]S \subseteq T$  and  $[\psi?]\psi \in S$ ,  $\psi \in T$ . Since  $P(\psi)$ ,  $T \in (\psi)^{\mathcal{M}_c}$ . Since  $S = T$ ,  $S(\psi?)^{\mathcal{M}_c} T$ .

**Case ‘ $\alpha = s_1$ ’.** Let  $S, T$  be maximal consistent theories.

- Suppose  $S(s_1)^{\mathcal{M}_c} T$ . We demonstrate there exists a maximal program  $A$  such that  $s_1 \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . Since  $S(s_1)^{\mathcal{M}_c} T$ , there exists a maximal consistent theory  $U$  such that  $T \in S \star_c U$ . Hence,  $[s_1]S \subseteq T$  and  $[s_2]U \subseteq T$ . Thus, it suffices to take  $A = s_1$ .
- Suppose there exists a maximal program  $A$  such that  $s_1 \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . We demonstrate  $S(s_1)^{\mathcal{M}_c} T$ . Since  $s_1 \in \ker(A)$ ,  $A = s_1$ . Since for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ ,  $[s_1]S \subseteq T$ . By (A4),  $[r_1]T \subseteq S$ . Hence,  $\langle r_1 \rangle \top \in T$ . By (A8),  $\langle r_2 \rangle \top \in T$ . By Lemma 10, there exists a maximal program  $B$  and there exists a maximal consistent theory  $U$  such that  $r_2 \in \ker(B)$  and for all programs  $\gamma$ , if  $\gamma \in \ker(B)$ ,  $[\gamma]T \subseteq U$ . Thus,  $B = r_2$ . Since for all programs  $\gamma$ , if  $\gamma \in \ker(B)$ ,  $[\gamma]T \subseteq U$ ,  $[r_2]T \subseteq U$ . By (A7),  $[s_2]U \subseteq T$ . Since  $[s_1]S \subseteq T$ ,  $T \in S \star_c U$ . Therefore,  $S(s_1)^{\mathcal{M}_c} T$ .

**Case ‘ $\alpha = s_2$ ’.** Similar to the case ‘ $\alpha = s_1$ ’.

**Case ‘ $\alpha = r_1$ ’.** Let  $S, T$  be maximal consistent theories.

- Suppose  $S(r_1)^{\mathcal{M}_c} T$ . We demonstrate there exists a maximal program  $A$  such that  $r_1 \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . Since  $S(r_1)^{\mathcal{M}_c} T$ , there exists a maximal consistent theory  $U$  such that  $S \in T \star_c U$ . Hence,  $[s_1]T \subseteq S$  and  $[s_2]U \subseteq S$ . By (A4),  $[r_1]S \subseteq T$ . Thus, it suffices to take  $A = r_1$ .
- Suppose there exists a maximal program  $A$  such that  $r_1 \in \ker(A)$  and for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ . We demonstrate  $S(r_1)^{\mathcal{M}_c} T$ . Since  $r_1 \in \ker(A)$ ,  $A = r_1$ . Since for all programs  $\beta$ , if  $\beta \in \ker(A)$ ,  $[\beta]S \subseteq T$ ,  $[r_1]S \subseteq T$ . By (A6),  $[s_1]T \subseteq S$ . Since  $[r_1]S \subseteq T$ ,  $\langle r_1 \rangle \top \in S$ . By (A8),  $\langle r_2 \rangle \top \in S$ . By Lemma 10, there exists a maximal program  $B$  and there exists a maximal consistent theory  $U$  such that  $r_2 \in \ker(B)$  and for all programs  $\gamma$ , if  $\gamma \in \ker(B)$ ,  $[\gamma]S \subseteq U$ . Hence,  $B = r_2$ . Since for all programs  $\gamma$ , if  $\gamma \in \ker(B)$ ,  $[\gamma]S \subseteq U$ ,  $[r_2]S \subseteq U$ . By (A7),  $[s_2]U \subseteq S$ . Since  $[s_1]T \subseteq S$ ,  $S \in T \star_c U$ . Therefore,  $S(r_1)^{\mathcal{M}_c} T$ .

**Case ‘ $\alpha = r_2$ ’.** Similar to the case ‘ $\alpha = r_1$ ’.

**Case ‘ $\alpha = \beta; \gamma$ ’.** Let  $S, T$  be maximal consistent theories.

- Suppose  $S(\beta; \gamma)^{\mathcal{M}_c} T$ . We demonstrate there exists a maximal program  $A$  such that  $f(\beta); \top?; f(\gamma) \in \ker(A)$  and for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ . Since  $S(\beta; \gamma)^{\mathcal{M}_c} T$ , there exists a maximal consistent theory  $U$  such that  $S(\beta)^{\mathcal{M}_c} U$  and  $U(\gamma)^{\mathcal{M}_c} T$ . Since  $P(\beta)$  and  $P(\gamma)$ , there exists a maximal program  $A'$  such that  $f(\beta) \in \ker(A')$  and for all programs  $\delta'$ , if  $\delta' \in \ker(A')$ ,  $[\delta']S \subseteq U$  and there exists a maximal program  $A''$  such that  $f(\gamma) \in \ker(A'')$  and for all programs  $\delta''$ , if  $\delta'' \in \ker(A'')$ ,  $[\delta'']U \subseteq T$ . Since  $\top \in U$ ,  $f(\beta); \top?; f(\gamma) \in \ker(A'; \bar{U}; A'')$ . Now, let  $\delta'; \phi?; \delta'' \in \ker(A'; \bar{U}; A'')$  and  $\psi \in [\delta'; \phi?; \delta'']S$ . Hence,  $\delta' \in \ker(A')$ ,  $\phi \in U$ ,  $\delta'' \in \ker(A'')$  and  $[\delta'; \phi?; \delta'']\psi \in S$ . Thus,  $[\delta'](\phi \rightarrow [\delta'']\psi) \in S$ . Therefore,  $\phi \rightarrow [\delta'']\psi \in [\delta']S$ . Since  $\delta' \in \ker(A')$ ,  $[\delta']S \subseteq U$ . Since  $\phi \rightarrow [\delta'']\psi \in [\delta']S$ ,  $\phi \rightarrow [\delta'']\psi \in U$ . Since  $\phi \in U$ ,  $[\delta'']\psi \in U$ . Consequently,  $\psi \in [\delta'']U$ . Since  $\delta'' \in \ker(A'')$ ,  $[\delta'']U \subseteq T$ . Since  $\psi \in [\delta'']U$ ,  $\psi \in T$ . Hence, for all programs  $\delta$ , if  $\delta \in \ker(A'; \bar{U}; A'')$ ,  $[\delta]S \subseteq T$ . Since  $f(\beta); \top?; f(\gamma) \in \ker(A'; \bar{U}; A'')$ , it suffices to take  $A = A'; \bar{U}; A''$ .
- Suppose there exists a maximal program  $A$  such that  $f(\beta); \top?; f(\gamma) \in \ker(A)$  and for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ . We demonstrate  $S(\beta; \gamma)^{\mathcal{M}_c} T$ . Since  $f(\beta); \top?; f(\gamma) \in \ker(A)$ , there exists a maximal program  $A'$ , there exists a maximal consistent theory  $U$  and there exists a maximal program  $A''$  such that  $f(\beta) \in \ker(A')$ ,  $f(\gamma) \in \ker(A'')$  and  $A = A'; \bar{U}; A''$ . Now, let  $\delta' \in \ker(A')$  and  $\phi \in [\delta']S$ . Hence,  $[\delta']\phi \in S$ . Let  $\delta'' \in \ker(A'')$ . Since  $[\delta']\phi \in S$ ,  $[\delta'](\neg\phi \rightarrow [\delta'']\perp) \in S$ . Thus,  $[\delta'; \neg\phi?; \delta'']\perp \in S$ . Therefore,  $\perp \in [\delta'; \neg\phi?; \delta'']S$ . Since  $T$  is consistent,  $\perp \notin T$ . Since for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$  and  $\perp \in [\delta'; \neg\phi?; \delta'']S$ ,  $\delta'; \neg\phi?; \delta'' \notin \ker(A)$ . Since  $A = A'; \bar{U}; A''$ ,  $\delta' \in \ker(A')$  and  $\delta'' \in \ker(A'')$ ,  $\neg\phi \notin U$ . Since  $U$  is maximal,  $\phi \in U$ . Consequently, for all  $\delta' \in \ker(A')$ ,  $[\delta']S \subseteq U$ . Since  $f(\beta) \in \ker(A')$  and  $P(\beta)$ ,  $S(\beta)^{\mathcal{M}_c} U$ . Now, let  $\delta'' \in \ker(A'')$  and  $\phi \in [\delta'']U$ . Hence,  $[\delta'']\phi \in U$ . Let  $\delta' \in \ker(A')$ . Thus,  $[\delta']([\delta'']\phi \rightarrow [\delta'']\phi) \in S$ . Therefore,  $[\delta'; [\delta'']\phi?; \delta'']\phi \in S$ . Consequently,  $\phi \in [\delta'; [\delta'']\phi?; \delta'']S$ . Since  $\delta' \in \ker(A')$ ,  $[\delta']\phi \in U$  and  $\delta'' \in \ker(A'')$ ,  $\delta'; [\delta'']\phi?; \delta'' \in \ker(A'; \bar{U}; A'')$ . Since  $A = A'; \bar{U}; A''$ ,  $\delta'; [\delta'']\phi?; \delta'' \in \ker(A)$ . Since for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ ,  $\delta'; [\delta'']\phi?; \delta'' \in \ker(A)$  and  $\phi \in [\delta'; [\delta'']\phi?; \delta'']S$ ,  $\phi \in T$ . Hence, for all  $\delta'' \in \ker(A'')$ ,  $[\delta'']U \subseteq T$ . Since  $f(\gamma) \in \ker(A'')$  and  $P(\gamma)$ ,  $U(\gamma)^{\mathcal{M}_c} T$ . Since  $S(\beta)^{\mathcal{M}_c} U$ ,  $S(\beta; \gamma)^{\mathcal{M}_c} T$ .

**Case ‘ $\alpha = \beta \parallel \gamma$ ’.** Let  $S, T$  be maximal consistent theories.

- Suppose  $S(\beta \parallel \gamma)^{\mathcal{M}_c} T$ . We demonstrate there exists a maximal program  $A$  such that  $(\top?; f(\beta); \top?) \parallel (\top?; f(\gamma); \top?) \in \ker(A)$  and for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ . Since  $S(\beta \parallel \gamma)^{\mathcal{M}_c} T$ , there exists maximal consistent theories  $U', U'', V', V''$  such that  $S \in U' \star_c U''$ ,  $T \in V' \star_c V''$ ,  $U'(\beta)^{\mathcal{M}_c} V'$  and  $U''(\gamma)^{\mathcal{M}_c} V''$ . Since  $P(\beta)$  and  $P(\gamma)$ , there exists a maximal program  $A'$  such that  $f(\beta) \in \ker(A')$  and for all programs  $\delta'$ , if  $\delta' \in \ker(A')$ ,  $[\delta']U' \subseteq V'$  and there exists a maximal program  $A''$  such that  $f(\gamma) \in \ker(A'')$  and for all programs  $\delta''$ , if  $\delta'' \in \ker(A'')$ ,  $[\delta'']V' \subseteq V''$ . Since  $\top \in U', U'', V', V''$ ,  $(\top?; f(\beta); \top?) \parallel (\top?; f(\gamma); \top?) \in \ker((\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}''))$ . Now, let  $(\phi'?; \delta'; \psi'??) \parallel (\phi''?; \delta''; \psi''?) \in \ker((\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}''))$  and  $\chi \in [(\phi'?; \delta'; \psi'??) \parallel (\phi''?; \delta''; \psi''?)]S$ . Hence,  $\phi' \in U'$ ,  $\delta' \in \ker(A')$ ,  $\psi' \in V'$ ,  $\phi'' \in U''$ ,  $\delta'' \in \ker(A'')$ ,  $\psi'' \in V''$  and  $[(\phi'?; \delta'; \psi'??) \parallel (\phi''?; \delta''; \psi''?)]\chi \in S$ . Thus,  $\langle(\phi'?; \delta'; \psi'??) \parallel (\phi''?; \delta''; \psi''?)\rangle \neg \chi \notin S$ . Since  $S$  is closed under the rule of proof (*FOR*), there exists an atomic formula  $p$  not occurring in  $\phi', \delta', \psi', \phi'', \delta'', \psi'', \chi$  such that  $\langle r_1 \rangle \langle \phi'?; \delta'; \psi'?? \rangle \langle s_1 \rangle (\neg \chi \wedge p) \vee \langle r_2 \rangle \langle \phi''?; \delta''; \psi''? \rangle \langle s_2 \rangle (\neg \chi \wedge \neg p) \notin S$ . Therefore,  $\langle r_1 \rangle \langle \phi'?; \delta'; \psi'?? \rangle \langle s_1 \rangle (\neg \chi \wedge p) \notin S$  and  $\langle r_2 \rangle \langle \phi''?; \delta''; \psi''? \rangle \langle s_2 \rangle (\neg \chi \wedge \neg p) \notin S$ . Consequently,  $[r_1][\phi'?; \delta'; \psi'??][s_1](\chi \vee \neg p) \in S$  and  $[r_2][\phi''?; \delta''; \psi''?][s_2](\chi \vee p) \in S$ . Hence,  $[r_1](\phi' \rightarrow [\delta'](\psi' \rightarrow [s_1](\chi \vee \neg p))) \in S$  and  $[r_2](\phi'' \rightarrow [\delta''](\psi'' \rightarrow [s_2](\chi \vee p))) \in S$ . Since  $S \in U' \star_c U''$ ,  $[s_1]U' \subseteq S$  and  $[s_2]U'' \subseteq S$ . By (A4) and (A5),  $[r_1]S \subseteq U'$  and  $[r_2]S \subseteq U''$ . Since  $[r_1](\phi' \rightarrow [\delta'](\psi' \rightarrow [s_1](\chi \vee \neg p))) \in S$  and  $[r_2](\phi'' \rightarrow [\delta''](\psi'' \rightarrow [s_2](\chi \vee p))) \in S$ ,  $\phi' \rightarrow [\delta'](\psi' \rightarrow [s_1](\chi \vee \neg p)) \in U'$  and  $\phi'' \rightarrow [\delta''](\psi'' \rightarrow [s_2](\chi \vee p)) \in U''$ . Since  $\phi' \in U'$  and  $\phi'' \in U''$ ,  $[\delta'](\psi' \rightarrow [s_1](\chi \vee \neg p)) \in U'$  and  $[\delta''](\psi'' \rightarrow [s_2](\chi \vee p)) \in U''$ . Since  $\delta' \in \ker(A')$  and  $\delta'' \in \ker(A'')$ ,  $[\delta']U' \subseteq U'$  and  $[\delta'']V' \subseteq V''$ . Since  $[\delta'](\psi' \rightarrow [s_1](\chi \vee \neg p)) \in U'$  and  $[\delta''](\psi'' \rightarrow [s_2](\chi \vee p)) \in U''$ ,  $\psi' \rightarrow [s_1](\chi \vee \neg p) \in V'$  and  $\psi'' \rightarrow [s_2](\chi \vee p) \in V''$ . Since  $\psi' \in V'$  and  $\psi'' \in V''$ ,  $[s_1](\chi \vee \neg p) \in V'$  and  $[s_2](\chi \vee p) \in V''$ . Since  $T \in V' \star_c V''$ ,  $[s_1]V' \subseteq T$  and  $[s_2]V'' \subseteq T$ . Since  $[s_1](\chi \vee \neg p) \in V'$  and  $[s_2](\chi \vee p) \in V''$ ,  $\chi \vee \neg p \in T$  and  $\chi \vee p \in T$ . Thus,  $\chi \in T$ . Therefore, for all programs  $\delta$ , if  $\delta \in \ker((\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}''))$ ,  $[\delta]S \subseteq T$ . Since  $(\top?; f(\beta); \top?) \parallel (\top?; f(\gamma); \top?) \in \ker((\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}''))$ , it suffices to take  $A = (\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}'')$ .
- Suppose there exists a maximal program  $A$  such that  $(\top?; f(\beta); \top?) \parallel (\top?; f(\gamma); \top?) \in \ker(A)$  and for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ . We demonstrate  $S(\beta \parallel \gamma)^{\mathcal{M}_c} T$ . Since  $(\top?; f(\beta); \top?) \parallel (\top?; f(\gamma); \top?) \in \ker(A)$ , there exists maximal consistent theories  $U', U'', V', V''$  and there exists maximal programs  $A', A''$  such that  $f(\beta) \in \ker(A')$ ,  $f(\gamma) \in \ker(A'')$  and  $A = (\bar{U}', ; A'; \bar{V}') \parallel (\bar{U}''; A''; \bar{V}'')$ . Let  $\phi \in [s_1]U'$ . Hence,  $[s_1]\phi \in U'$ . Since for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ ,  $\langle([s_1]\phi?; \delta'; \top?) \parallel (\top?; \delta''; \top?)\rangle \top \in S$  for each program  $\delta' \in \ker(A')$  and for each program  $\delta'' \in \ker(A'')$ . By (A10),  $\langle r_1 \rangle \langle [s_1]\phi?; \delta'; \top? \rangle \langle s_1 \rangle (\top \wedge \top) \vee \langle r_2 \rangle \langle \top?; \delta''; \top? \rangle \langle s_2 \rangle (\top \wedge \neg \top) \in S$ . Thus,  $\langle r_1 \rangle \langle [s_1]\phi?; \delta'; \top? \rangle \langle s_1 \rangle \top \in S$ . Therefore,  $\langle r_1 \rangle [s_1]\phi \in S$ . By (A6),  $\phi \in S$ . Consequently,  $[s_1]U' \subseteq S$ . The argument showing  $[s_2]U'' \subseteq S$  is obtained in a similar way. Hence,  $S \in U' \star_c U''$ . Let  $\phi \in [s_1]V'$ . Hence,  $[s_1]\phi \in V'$ . Suppose  $\phi \notin T$ . Thus,  $\neg \phi \in T$ . Since for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ ,  $\langle(\top?; \delta'; [s_1]\phi?) \parallel (\top?; \delta''; \top?)\rangle \neg \phi \in S$  for each program  $\delta' \in \ker(A')$  and for each program  $\delta'' \in \ker(A'')$ . By (A10),  $\langle r_1 \rangle \langle \top?; \delta'; [s_1]\phi? \rangle \langle s_1 \rangle (\neg \phi \wedge \top) \vee \langle r_2 \rangle \langle \top?; \delta''; \top? \rangle \langle s_2 \rangle (\neg \phi \wedge \neg \top) \in S$ . Thus,  $\langle r_1 \rangle \langle \top?; \delta'; [s_1]\phi? \rangle \langle s_1 \rangle \neg \phi \in S$ . Therefore,  $\langle r_1 \rangle \langle \delta' \rangle ([s_1]\phi \wedge [s_1]\neg \phi) \in S$ : a contradiction with the consistency of  $S$ . Consequently,  $\phi \in T$ . Hence,  $[s_1]V' \subseteq T$ . The argument showing  $[s_2]V'' \subseteq T$  is obtained in a similar way. Hence,  $T \in V' \star_c V''$ . Let  $\delta' \in \ker(A')$  and  $\phi \in [\delta']U'$ . Thus,  $[\delta']\phi \in U'$ . Suppose  $\phi \notin V'$ . Therefore,  $\neg \phi \in V'$ . Since for all programs  $\delta$ , if  $\delta \in \ker(A)$ ,  $[\delta]S \subseteq T$ ,  $\langle([\delta']\phi?; \delta'; \neg \phi?) \parallel (\top?; \delta''; \top?)\rangle \top \in S$  for each program  $\delta'' \in \ker(A'')$ . By (A10),  $\langle r_1 \rangle \langle [\delta']\phi?; \delta'; \neg \phi? \rangle \langle s_1 \rangle (\top \wedge \top) \vee \langle r_2 \rangle \langle \top?; \delta''; \top? \rangle \langle s_2 \rangle (\top \wedge \neg \top) \in S$ . Consequently,  $\langle r_1 \rangle \langle [\delta']\phi?; \delta'; \neg \phi? \rangle \langle s_1 \rangle \top \in S$ . Hence,  $\langle r_1 \rangle ([\delta']\phi \wedge \neg \phi) \in S$ : a contradiction with the



consistency of  $S$ . Thus,  $\phi \in V'$ . Therefore, for all  $\delta' \in \ker(A')$ ,  $[\delta']U' \subseteq V'$ . The argument showing for all  $\delta'' \in \ker(A'')$ ,  $[\delta'']U'' \subseteq V''$  is obtained in a similar way. Since  $f(\beta) \in \ker(A')$  and  $f(\gamma) \in \ker(A'')$ , by induction hypothesis,  $U'(\beta)^{\mathcal{M}_c} V'$  and  $U''(\gamma)^{\mathcal{M}_c} V''$ . Since  $S \in U' \star_c U''$  and  $T \in V' \star_c V''$ ,  $S(\beta \parallel \gamma)^{\mathcal{M}_c} T$ .

**Case ‘ $\phi = p$ ’.** Left to the reader.

**Cases ‘ $\phi = \perp$ ’, ‘ $\phi = \neg\psi$ ’ and ‘ $\phi = \psi \vee \chi$ ’.** Left to the reader.

**Case ‘ $\phi = [\beta]\psi$ ’.** Let  $S$  be a maximal consistent theory.

- Suppose  $S \in ([\beta]\psi)^{\mathcal{M}_c}$ . We demonstrate  $[\beta]\psi \in S$ . If not, by Lemma 10, there exists a maximal program  $A$  and there exists a maximal consistent theory  $T$  such that  $f(\beta) \in \ker(A)$ , for all programs  $\gamma$ , if  $\gamma \in \ker(A)$ ,  $[\gamma]S \subseteq T$  and  $\psi \notin T$ . Since  $P(\beta)$  and  $P(\psi)$ ,  $S(\beta)^{\mathcal{M}_c} T$  and  $T \notin (\psi)^{\mathcal{M}_c}$ . Hence,  $S \notin ([\beta]\psi)^{\mathcal{M}_c}$ : a contradiction.
- Suppose  $[\beta]\psi \in S$ . We demonstrate  $S \in^{\mathcal{M}_c} ([\beta]\psi)^{\mathcal{M}_c}$ . If not, there exists a maximal consistent theory  $T$  such that  $S(\beta)^{\mathcal{M}_c} T$  and  $T \notin (\psi)^{\mathcal{M}_c}$ . Since  $P(\beta)$  and  $P(\psi)$ , there exists a maximal program  $A$  such that  $f(\beta) \in \ker(A)$  and for all programs  $\gamma$ , if  $\gamma \in \ker(A)$ ,  $[\gamma]S \subseteq T$  and  $\psi \notin T$ . Hence,  $[f(\beta)]S \subseteq T$ . By (A12), since  $[\beta]\psi \in S$ ,  $[f(\beta)]\psi \in S$ . Thus,  $\psi \in [f(\beta)]S$ . Since  $[f(\beta)]S \subseteq T$ ,  $\psi \in T$ : a contradiction.

■

And now, the grand finale:

**PROPOSITION 12 (Completeness for  $PRSPDL_0$ )**

Let  $\phi$  be a formula. If  $\phi$  is valid in the class of all separated frames,  $\phi \in PRSPDL_0$ .

**PROOF.** By Lemmas 5, 7, 8 and 11.

■

## 8 Conclusion

We have given a complete axiomatization of iteration-free  $PDL$  with storing, recovering and parallel composition. Because of the subtleties brought about by the construct  $(\cdot \parallel \cdot)$  of parallel composition, we have used maximal programs in the proofs of the Existence Lemma and the Truth Lemma of the canonical model construction.

Although we know that validity in the class of all separated frames is  $\Pi_1^1$ -complete when the construct  $\star$  of iteration is added to the language [2], we expect that maximal programs can also be applied for proving the completeness of an axiomatization of the full version of  $PRSPDL$ . Remind that after [5], the problem of finding a complete axiomatization of  $PRSPDL$  remained open. We believe our complete axiomatization of  $PRSPDL_0$  constitutes a first step in the direction of an axiomatization of the full version of  $PRSPDL$ .

Another issue concerns the complete axiomatization of  $PRSPDL$  when parallel composition  $(\alpha \parallel \beta)$  of programs  $\alpha$  and  $\beta$  is interpreted in such a way that state  $x$  and states in  $y \ast z$  are related via  $R(\alpha) \nabla R(\beta)$  whenever  $x$  and  $y$  are related via  $R(\alpha)$  and  $x$  and  $z$  are related via  $R(\beta)$ . See [13, Chapter 1] for such an interpretation.

But the general problem that remains open is the following: is it possible to replace the rule of proof ( $FOR$ ) by finitely many additional axiom schemes? The solution to a similar problem about iteration-free  $PDL$  with intersection given in [1] has revealed interesting validities like formulas of the form  $[(\alpha; (\psi \wedge [(\beta; \phi?; \alpha) \cap \psi?]) \perp); \beta] \cap \phi? \perp$ . We believe that the elimination of the rule of proof ( $FOR$ ) from our axiom system for iteration-free  $PRSPDL$  could reveal similar interesting validities.



Finally, Proposition 8 implies that tests cannot be defined in terms of the other constructs of the language of  $PRSPDL_0$ . Within the context of the language of  $PDL$ , a similar result has been generalized in [6] where a strict hierarchy  $PDL_0 \subset PDL_1 \subset \dots$  of fragments of the language of  $PDL$  has been defined in such a way that for all non-negative integers  $n$ , a test  $\phi?$  is permitted to occur in a formula of  $PDL_n$  only if  $\phi$  belongs to  $PDL_0 \cup \dots \cup PDL_{n-1}$ . We believe that Proposition 8 can be generalized in a similar way.

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